

WARPED PRODUCTS ADMITTING A CURVATURE BOUND

STEPHANIE B. ALEXANDER AND RICHARD L. BISHOP

ABSTRACT. This paper completes a fundamental construction in Alexandrov geometry. Previously we gave a new construction of metric spaces with curvature bounds either above or below, namely warped products with intrinsic metric space base and fiber, and with possibly vanishing warping functions – thereby extending the classical cone and suspension constructions from interval base to arbitrary base, and furthermore encompassing gluing constructions. This paper proves the converse, namely, all conditions of the theorems are necessary. Note that in the cone construction, both the construction and its converse are widely used. We also show that our theorems for curvature bounded above and below, respectively, are dual. We give the first systematic development of basic properties of warped products of metric spaces with possibly vanishing warping functions, including new properties.

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1. INTRODUCTION

This paper completes a fundamental construction in the theory of Alexandrov spaces. In classical Riemannian geometry, warped products provide perhaps the major source of examples and counter-examples. In [AB 04], a new construction of metric spaces with curvature bounds either above or below was given, namely warped products with intrinsic metric space base and fiber, and with possibly vanishing warping functions – thereby extending the classical cone and suspension

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constructions from interval base to arbitrary base, and furthermore encompassing gluing constructions.

This paper proves the converse, namely, all the conditions of the theorems are necessary. Note that in the cone construction, the implications in both directions are widely used.

Among the rather delicate arguments required, it turned out that adequate tools to handle all of them were not available when [AB 04] was written. For spaces of curvature bounded below, we know of no proof that does not use Petrunin's globalization theorem for incomplete spaces [Pt 12]. We use it to prove a gluing theorem on the closure of the subset of the boundary on which the warping function is nonvanishing. For curvature bounded above, to prove the correct bound on the fiber we had first to obtain a Gauss Lemma for curvature of general subspaces [AB 06].

In [AB 04], we did not recognize that our theorems for curvature bounded above and below, respectively, were completely dual. Here we show this by proving equivalent formulations of the original hypotheses.

We also give a systematic development, including new properties, of warped products of metric spaces with somewhere-vanishing warping functions. These were introduced in [AB 04], where their treatment was *ad hoc*. Somewhere-vanishing warping functions greatly enrich our source of examples and counter-examples by allowing gluing on subsets.

The proofs given here illustrate a range of techniques and constructions in Alexandrov geometry. We try to bring into focus the dualities between curvature bounded below and above

2. STATEMENT OF THEOREMS

Let $(X, |**| : X \times X \rightarrow [0, \infty))$ be a metric space. The *model angle* $\tilde{\angle}^\kappa[x^i <_{x^k} x^j]$ is the angle corresponding to x^i in the *model triangle* $\tilde{\Delta}^\kappa[x^i x^j x^k]$ with sidelengths $|x^i x^j|, |x^j x^k|, |x^k x^i|$, in the complete simply connected surface of constant curvature κ . We call that surface the *model surface* for κ . The model triangle and model angle are said to be *defined* if there is a unique triangle in the model surface with those sidelengths. In particular, the perimeter of the model triangle is $\leq \varpi^\kappa = \pi/\sqrt{\kappa}$ ($= \infty$ if $\kappa \leq 0$).

As is well known, $X \in \text{CBB}^\kappa$ and $X \in \text{CAT}^\kappa$ may be defined using *point-side κ -comparisons*. Namely, for every point x^1 and geodesic $[x^2 x^3]$ such that $\tilde{\Delta}^\kappa[x^1 x^2 x^3]$ is defined, the distance between x^1 and each point of $[x^2 x^3]$ is \geq (for CBB^κ) or \leq (for CAT^κ) the distance between the corresponding points of $\tilde{\Delta}^\kappa[x^1 x^2 x^3]$.

However, we are going to use instead, equivalent definitions that depend on distance only. Our CAT^κ definition is new in [AKP]. See Section 3.1.

Let f be a locally Lipschitz function defined on a metric space. For $\kappa \in \mathbf{R}$, we say f is *sinusoidally κ -convex*, written $f \in \tilde{\mathcal{C}}^\kappa$, if for every unit-speed geodesic γ ,

$$(f \circ \gamma)'' + \kappa \cdot (f \circ \gamma) \geq 0.$$

If the inequality is reversed, we say f is *sinusoidally κ -concave*, written $f \in \hat{\mathcal{C}}^\kappa$. The inequalities are meant in the generalized sense: if $y'' + \kappa \cdot y = 0$, and y and $(f \circ \gamma)$ are defined on and coincide at the endpoints of a sufficiently short interval, then $(f \circ \gamma) \leq y$ (respectively $\geq y$) That is, there are *two-point supports*. Equivalently, at every point there is a solution y of $y'' + \kappa \cdot y = 0$ defined on an open interval,

coinciding with $f \circ \gamma$ at that point, and satisfying the opposite inequality – *tangential supports* exist. Thus sinusoidal 0-convexity (0-concavity) is convexity (concavity) in the usual sense.

Definition 2.1. Let B and F be intrinsic spaces, and $f : B \rightarrow \mathbf{R}_{\geq 0}$ be locally Lipschitz. Suppose $F \neq \text{point}$, and $Z = f^{-1}(0) \neq B$. When we denote a warped product by $B \times_f F$, we assume (B, f, F) is such a triple, which we call a *WP-triple*.

Theorem 2.2 (CAT^κ). *Let (B, f, F) be a WP-triple, and assume f is Lipschitz on bounded sets. Then $B \times_f F \in \text{CAT}^\kappa$ if and only if the following conditions hold, where $Z = f^{-1}(0)$:*

- (1) $B \in \text{CAT}^\kappa$ and $f \in \check{\mathcal{C}}^\kappa$.
- (2) If $Z = \emptyset$, then $F \in \text{CAT}^{\kappa_F}$ for

$$\kappa_F = \kappa \cdot (\inf f)^2.$$
- (3) If $Z \neq \emptyset$, then $F \in \text{CAT}^{\kappa_F}$ for $\kappa_F = \min \{\kappa_{\text{foot}}, \kappa_{\text{far}}\}$, where

$$\kappa_{\text{foot}} = \inf \{(f \circ \alpha)^+(0)^2 : \alpha = \text{dist}_Z\text{-realizer with footpoint } \alpha(0) \in Z, |\alpha^+(0)| = 1\},$$

$$\kappa_{\text{far}} = \inf \{\kappa \cdot f(p)^2 : \text{dist}_Z(p) \geq \varpi^\kappa/2\}.$$

Theorem 2.3. (CBB^κ) *Let (B, f, F) be a WP-triple, and assume f is Lipschitz on bounded sets. Then $B \times_f F \in \text{CBB}^\kappa$ if and only if the following conditions hold, where $Z = f^{-1}(0)$:*

- (1) $B \in \text{CBB}^\kappa$ and $f \in \widehat{\mathcal{C}}^\kappa$.
- (2) Let $B^\dagger(f)$ be obtained by gluing two copies of B on $\text{closure}(\partial B - Z)$, and let $f^\dagger : B^\dagger \rightarrow [0, \infty)$ be the tautological extension of f . Then $B^\dagger(f) \in \text{CBB}^\kappa$ and $f^\dagger \in \widehat{\mathcal{C}}^\kappa$.
- (3) If $Z = \emptyset$, then $F \in \text{CBB}^{\kappa_F}$ for

$$\kappa_F = \kappa \cdot (\inf f)^2.$$
- (4) If $Z \neq \emptyset$, then $F \in \text{CBB}^{\kappa_F}$ for

$$\kappa_F = \sup \{(f \circ \alpha)^+(0)^2 : \alpha = \text{dist}_Z\text{-realizer with footpoint } \alpha(0) \in Z, |\alpha^+(0)| = 1\}.$$

Theorem 2.4. (a) In Theorem 2.2 (3), we may substitute

$$\kappa_{\text{foot}} = \liminf_{\epsilon \rightarrow 0} \{|\nabla_p(-f)|^2 : 0 < \text{dist}_Z(p) \leq \epsilon\}.$$

(b) In Theorem 2.3 (4), we may substitute

$$\kappa_F = \sup \{|\nabla_q f|^2 : q \in Z\}.$$

Remarks 2.5. (a) In Theorem 2.2, condition (1) implies Z is ϖ^κ -convex. In Theorem 2.3, condition (1) implies $Z \subset \partial B$.

(b) In Theorem 2.3 (3) we may substitute

$$\kappa_F \geq \kappa \cdot f^2.$$

This is because when $Z = \emptyset$, conditions (1) and (2) imply $\kappa \leq 0$ (see proof of Lemma 7.2).

(c) Theorems 2.2(2) and 2.3(3) are asymptotic versions of a basic fact for warped products: if f achieves a positive minimum at p_0 , then shorter joins between points of $\{p_0\} \times F$ cannot be achieved by leaving $\{p_0\} \times F$.

(d) For the simple example

$$\text{Cone } F = \mathbf{R}_{\geq 0} \times_{\text{id}} F,$$

Theorems 2.2 and 2.3 reduce to the well-known statements: $\text{Cone } F \in \text{CAT}^0$ if and only if $F \in \text{CAT}^1$; $\text{Cone } F \in \text{CBB}^0$ if and only if $F \in \text{CBB}^1$. (See [BBI 01, Theorem 4.7.1].) If we allow F to be a disjoint union of intrinsic spaces F_α , then in the first statement substitute: each component $F_\alpha \in \text{CAT}^1$. In the second statement, substitute: $F \in \text{CBB}^1$ or $F = 2$ points. Here, $F = 2$ points is the only additional possibility since otherwise geodesic bifurcations occur at the vertex.

3. BACKGROUND AND CONVENTIONS

Here we summarize our tools:

3.1. Curvature bounds. Definitions and basic theorems are discussed in [BBI 01] and [AKP, *Definitions of CBB/CBA*]; and in [BGP 92], [Pl 02], [S 93] for CBB^κ , and [BH 99] for CAT^κ .

Given a metric space X , we are going to use the following definitions.

- A *geodesic* γ joining $x^1, x^2 \in X$ is a constant-speed curve of length $|x^1 x^2|$. We may also denote γ by $[x^1 x^2]$. A *pregeodesic* is a monotonically reparametrized geodesic. A geodesic (pregeodesic) is said to be unique if it is determined by its endpoints up to reparametrization. X is *geodesic (intrinsic)* if any $x^1, x^2 \in X$ are joined by a geodesic (respectively, by curves of length arbitrarily close to $|x^1 x^2|$). X is *r-geodesic (r-intrinsic)* if this condition is applied only when $|x^1 x^2| < r$.
- A quadruple of points x^1, x^2, x^3, x^4 in a metric space satisfies $(1+3)$ -point κ -comparison, briefly $(1+3)^\kappa$, if

$$\tilde{\kappa}[x^1 \angle_{x^3} x^2] + \tilde{\kappa}[x^1 \angle_{x^4} x^3] + \tilde{\kappa}[x^1 \angle_{x^2} x^4] \leq 2 \cdot \pi,$$

or at least one of the three model angles $\tilde{\kappa}[x^1 \angle_{x^i} x^j]$ is undefined [BGP 92].

- A quadruple of points x^1, x^2, x^3, x^4 in a metric space satisfies $(2+2)$ -point κ -comparison, briefly $(2+2)^\kappa$, if
 - (a) either $\tilde{\kappa}[x^1 \angle_{x^4} x^3] \leq \tilde{\kappa}[x^1 \angle_{x^2} x^3] + \tilde{\kappa}[x^1 \angle_{x^4} x^2]$,
 - (b) or $\tilde{\kappa}[x^2 \angle_{x^4} x^3] \leq \tilde{\kappa}[x^2 \angle_{x^1} x^3] + \tilde{\kappa}[x^2 \angle_{x^4} x^1]$,
 or at least one of the six model angles $\tilde{\kappa}[x^i \angle_{x^k} x^j]$ is undefined [AKP].

The following definitions of CBB^κ and CAT^κ are equivalent to point-side definitions, but depend on distances only, not on existence of geodesics. They allow us to give some substantially simpler proofs.

Definition 3.1. Let X be an intrinsic space.

- $X \in \text{CBB}^\kappa$ means X is a complete intrinsic space in which every quadruple satisfies $(1+3)^\kappa$. In this paper, we further assume X has *finite dimension*. (In particular, X is proper, hence a geodesic space, and boundary ∂X is defined.) We also use the convention (for $\kappa > 0$) that X is not isometric to

a closed interval of length $> \varpi^\kappa$, or a circle of length $> 2 \cdot \varpi^\kappa$. Under this convention, if $X \in \text{CBB}^\kappa$, then X has diameter $\leq \varpi^\kappa$ [BGP 92].

- X has *curvature* $\geq \kappa$, written $\text{curv } X \geq \kappa$, if any point $p \in X$ has a neighborhood Ω_p such that all quadruples lying in Ω_p satisfy $(1+3)^\kappa$;
- [AKP] $X \in \text{CAT}^\kappa$ means X is a complete intrinsic space in which every quadruple satisfies $(2+2)^\kappa$. It follows that X is ϖ^κ -geodesic.
- X has *curvature* $\leq \kappa$, written $\text{curv } X \leq \kappa$, if any point $p \in X$ has a neighborhood Ω_p such that all quadruples lying in Ω_p satisfy $(2+2)^\kappa$.

3.2. Globalization. The following theorem [Pt 12] extends the Burago-Gromov-Perelman globalization theorem [BGP 92] to incomplete spaces. Theorem 3.2 includes long intervals and circles in CBB^κ , rather than using our convention which excludes them.

Theorem 3.2 (Petrinin's incomplete-globalization theorem [Pt 12]). *Let X be a geodesic space and \hat{X} be its completion. Suppose $\text{curv } X \geq \kappa$. Then $\hat{X} \in \text{CBB}^\kappa$.*

3.3. Definitions. Let X be a metric space. The *speed* of a curve $\alpha : J \rightarrow X$ at $t_0 \in J$, where J is an interval, is defined as

$$(\text{speed } \alpha)(t_0) = \lim_{t \rightarrow t_0, t \in J} (|\alpha(t) - \alpha(t_0)| / |t - t_0|).$$

If α is Lipschitz, then *speed* α exists at almost all $t \in J$, and *length* α is finite and given by Lebesgue integral of *speed*.

A subset $S \subset X$ will be called *convex in X* if all $x^1, x^2 \in S$ are joined by geodesics of X , and all such geodesics lie in S . If this condition holds when $|x^1 - x^2| < r$, then S is said to be *r -convex*.

For $S \subset X$, we denote distance from S by dist_S . Set

$$(3.1) \quad \mathcal{B}(S, r) = \{x \in X : \text{dist}_S(x) < r\}.$$

If $S = \{p\}$, we write $\mathcal{B}(p, r)$ for the open ball of radius r about p , and $\overline{\mathcal{B}}(p, r)$ for the closure of $\mathcal{B}(p, r)$.

3.4. Tangent spaces and differentials. Suppose $X \in \text{CBB}^\kappa$ or $X \in \text{CAT}^\kappa$. Recall that if γ^1 and γ^2 are geodesics from p , and x^i lies on γ^i , then $\tilde{Z}^\kappa[p \prec_{x^2}^{x^1}]$ is a monotone function of $(|p - x^1|, |p - x^2|)$. Then the angle at p between γ^1 and γ^2 is defined as

$$(3.2) \quad \lim_{|p - x^1| \rightarrow 0, |p - x^2| \rightarrow 0} \tilde{Z}^\kappa[p \prec_{x^2}^{x^1}].$$

Consider the set $\Gamma_p X$ of geodesics γ with $\gamma(0) = p$. Set $\gamma_1 \sim \gamma_2$ if γ_1 and γ_2 are non-constant and the angle between them is 0. A metric on the quotient space $(\Gamma_p X / \sim)$ is given by the angle between representative geodesics γ . We denote this metric space by $\Sigma'_p X$, the *space of geodesic directions*. The *space of directions* $\Sigma_p X$ is the completion of $\Sigma'_p X$.

The *tangent space*, or *space of tangent vectors*, $T_p X$, is the linear cone over $\Sigma_p X$:

$$T_p X = \text{Cone}(\Sigma_p X).$$

If γ is a geodesic with $\gamma(0) = p$ and *speed* $c > 0$, and u is the direction at p represented by γ , the *right derivative* $\gamma^+(0)$ of γ at 0 is the tangent “vector” $(c, u) \in T_p X$, which we write as $v = c \cdot u$. We denote the vertex of the cone $T_p X$ by o_p .

Let $f : X \rightarrow \mathbf{R}$ be a locally Lipschitz function such that $(f \circ \gamma)^+(0)$ exists for every geodesic γ with $\gamma(0) = p$. Then the *differential of f at p* is a uniquely determined, linearly homogeneous, Lipschitz map

$$d_p f : T_p X \rightarrow \mathbf{R}$$

such that $(d_p f)(x) = (f \circ \gamma)^+(0)$ when γ is a geodesic with $\gamma^+(0) = x$.

In this paper, convergence of spaces always refers to Gromov-Hausdorff convergence. We need:

Lemma 3.3 ([BGP 92, Pr 91]). *Suppose $X \in \text{CBB}^\kappa$. Then*

$$\begin{aligned} (T_p X, o_p) &= \lim_{\lambda \rightarrow \infty} (\lambda X, p), \\ (T_p X, \partial(T_p X), o_p) &= \lim_{\lambda \rightarrow \infty} (\lambda X, \partial(\lambda X), p) \text{ if } p \in \partial B. \end{aligned}$$

3.5. Convex functions and gradient vectors. Suppose a continuous function f on a metric space X is sinusoidally κ -concave or κ -convex, i.e. $f \in \hat{\mathcal{C}}^\kappa$ or $f \in \check{\mathcal{C}}^\kappa$. Then f is *semiconcave* (*semiconvex*), i.e. locally there is a constant generalized upper (lower) bound on f'' along unitspeed geodesics γ . Equivalently, $(f \circ \gamma)(t) - \lambda \cdot t^2$ is concave for some $\lambda \in \mathbf{R}$. The restriction of a semiconcave function f to a geodesic γ has all the regularity properties of a convex function: left and right derivatives exist at every point, and the second derivative exists almost everywhere.

Theorem 3.4. [Lt 05, Pt 06] *Let $X \in \text{CBB}^\kappa$ or $X \in \text{CAT}^\kappa$, and $f : X \rightarrow \mathbf{R}_{\geq 0}$ be a locally Lipschitz semiconcave function. Then:*

- (i) $d_p f$ exists and is concave.
- (ii) The gradient $\nabla_p f \in T_p X$ exists, where $\nabla_p f = o_p$ if $d_p f \leq 0$, and otherwise

$$\nabla_p f = (d_p f)(u_{\max}) \cdot u_{\max}$$

for the unique $u_{\max} \in \Sigma_p X$ at which $(d_p f)|_{\Sigma_p X}$ takes its maximum.

- (iii) *Maximal gradient curves, whose right tangents are everywhere equal to the gradient vector, exist and are unique.*

Proof. Gradient curves of semiconcave functions were introduced in [PP 94] (for $\text{curv} \geq \kappa$), and their properties developed by Lytchak [Lt 05] (for both $\text{curv} \geq \kappa$ and $\text{curv} \leq \kappa$) and Petrunin [Pt 06]. In the CAT^κ case, existence of the gradient vectors and gradient curves as defined here follows from [Lt 05] by invoking a Helly-type theorem (see [LS 97]). \square

Remark 3.5. When $X \in \text{CAT}^\kappa$, we are going to apply Theorem 3.4 to semiconvex functions f , by considering the gradient vectors and gradient curves of the semiconcave function $(-f)$. We call the gradient vectors $\nabla_p(-f)$ the *downward gradient vectors of f* , and the gradient curves of $-f$, the *downward gradient curves of f* .

Remark 3.6. In [AB 04, AB 96], sinusoidally κ -convex and κ -concave functions were called $\mathcal{F}\kappa$ -convex and $\mathcal{F}\kappa$ -concave.

4. WARPED PRODUCTS

Basic properties of warped products with *positive* warping functions, $f > 0$, were proved in [AB 98]. They were used in [AB 04], which treated vanishing of f in an *ad hoc* manner. In this paper we require a systematic treatment, including new properties, for $f \geq 0$, given in this section.

Let (B, f, F) be a WP-triple (Definition 2.1).

In this paper, J always denotes some finite closed interval.

Definition 4.1 (Warped product). Consider the topological space $(B \times F)/\sim$, where the elements of $\{p\} \times F$ are identified if $f(p) = 0$. We denote this class by \bar{p} , or by any of its representatives (p, φ) , $\varphi \in F$.

An *admissible curve* for the triple (B, f, F) is a curve $\gamma : J \rightarrow (B \times F)/\sim$. We write $\gamma = (\gamma_B, \gamma_F)$ for $\gamma_B : J \rightarrow B$ and $\gamma_F : J_+ \rightarrow F$, where $J_0 = (f \circ \gamma_B)^{-1}(0)$, $J_+ = J - J_0$. Set $J_+ = \sqcup_{i=1,2,\dots} J_i$, where the J_i are maximal open subintervals of J_+ . We further assume γ_B and $\gamma_F|_{J_i}$ to be Lipschitz, where $\gamma_F|_{J_i}$, $i = 1, \dots$, have a uniform Lipschitz constant. (Our class of admissible curves satisfies the concatenation property of length structures in the sense of [BBI 01]; see remark at end of this section.) Set

$$(4.1) \quad \text{length } \gamma = \int_J \sqrt{v_B^2 + (f \circ \gamma_B)^2 \cdot v_F^2},$$

where \int is Lebesgue integral, v_B is the speed of γ_B , $v_F|_{J_+}$ is the speed of $\gamma_F|_{J_+}$ and $v_F|_{J_0} = 0$. Then the integrand is defined almost everywhere on J and bounded.

Equivalently,

$$\text{length } \gamma = \sum_i \int_{J_i} \sqrt{v_B^2 + (f \circ \gamma_B)^2 \cdot v_F^2} + \text{length}(\gamma_B|_{J_0}).$$

Here the first term is defined, independently of enumeration, because the summands are positive.

Then the *warped product* $B \times_f F$ is the corresponding intrinsic space, where distance is given by infimum of lengths of admissible curves joining two given points.

We refer to B and F as *base* and *fiber* respectively. $B \times \{\varphi_0\}$ is called a *horizontal leaf*; and $\{p_0\} \times F$ when $f(p_0) > 0$, a *vertical leaf*.

Remark 4.2. The vanishing set $f^{-1}(0)$ of f represents the set on which the horizontal leaves $B \times \{\varphi_0\}$ are glued together. At these points there is no well-defined projection γ_F to F .

Proposition 4.3. *The warped product $B \times_f F$ satisfies:*

- (1) *The intrinsic and extrinsic metrics of any horizontal leaf $B \times \{\varphi_0\}$ agree, and projection $(p, \varphi_0) \mapsto p$ is an isometry onto B .*
- (2) *If $f(p_0) \neq 0$, then the projection $(p_0, \varphi) \mapsto \varphi$ of any vertical leaf $\{p_0\} \times F$, with its intrinsic metric, is a homothety onto F with multiplier $1/f(p_0)$.*
- (3) *If f achieves a positive minimum at p_0 , then the intrinsic and extrinsic metrics of $\{p_0\} \times F$ agree.*

Proof. Claims (1) and (2) are immediate from the length formula (4.1).

Also by (4.1), the projection onto $\{p_0\} \times F$ given by $(p, \varphi) \mapsto (p_0, \varphi)$ is length-nonincreasing if p_0 is a minimum point of f . Hence (3). \square

Remark 4.4. A horizontal leaf need not be convex even if $B \times_f F$ is a geodesic space, since vanishing of the warping function f allows geodesics to bifurcate into distinct horizontal leaves. For instance, suppose $\alpha : [0, 1] \rightarrow B$ is a geodesic of B such that $f(\alpha(0)) = f(\alpha(1)) = 0$ and $f \circ \alpha$ is not identically 0. Then for any distinct $\varphi_1, \varphi_2 \in F$, the geodesic (α, φ_2) of $B \times_f F$ has its endpoints in $B \times \{\varphi_1\}$ but does not lie in $B \times \{\varphi_1\}$.

Now we show that distance in a warped product is fiber-independent, in the sense that distances may be calculated by substituting for F a different intrinsic space. Propositions 4.3 and 4.5 summarize properties that for the case $f > 0$ are given in [AB 98]. Proposition 4.3(2) is due to Chen.

Proposition 4.5 (Fiber independence). *Let $W = B \times_f F$ and $W^* = B \times_f F^*$, where $F^* \neq \text{point}$ is an intrinsic space.*

- (1) *Let $p, q \in B$, $\varphi, \psi \in F$, and $\varphi^*, \psi^* \in F^*$.*

$$\text{If } |\varphi\psi|_F = |\varphi^*\psi^*|_{F^*}, \quad \text{then } |(p, \varphi)(q, \psi)|_W = |(p, \varphi^*)(q, \psi^*)|_{W^*}.$$

Let $\gamma = (\gamma_B, \gamma_F) : J \rightarrow W$ be a geodesic such that $f \circ \gamma_B > 0$:

- (2) *[Ch 99] γ_F is a pregeodesic in F .*
 (3) *Suppose $\beta^* : J \rightarrow F^*$ is a pregeodesic in F^* such that β^* and γ_F have the same speed, i.e. $v_{F^*} = v_F$ where v_{F^*} is the speed of β^* . Then (γ_B, β^*) is a geodesic in $B \times_f F^*$.*

Proof. Let $\gamma_i : J \rightarrow W$ be admissible curves with endpoints $(p, \varphi), (q, \psi)$, where $\text{length } \gamma_i \rightarrow |(p, \varphi)(q, \psi)|$.

Suppose $f \circ (\gamma_i)_B > 0$. Set $L_i = \text{length } (\gamma_i)_F$ and $v_i = \text{speed } (\gamma_i)_F$. Let β_i^* be curves in F^* with endpoints φ^*, ψ^* and lengths $L_i^* \rightarrow |\varphi^*\psi^*| = |\varphi\psi|$. Without loss of generality, $L_i^* \leq L_i + \epsilon_i$, $\epsilon_i \rightarrow 0$. Define $\gamma_i^* : J \rightarrow W^*$ by setting $(\gamma_i^*)_B = (\gamma_i)_B$, and letting $(\gamma_i^*)_{F^*}$ be β_i^* reparametrized with speed $(L_i^*/L_i) \cdot v_i$. By the length formula (4.1), $|(p, \varphi)(q, \psi)| = \lim \text{length}(\gamma_i) \geq \limsup(\text{length } \gamma_i^*)$. Hence $|(p, \varphi)(q, \psi)| \geq |(p, \varphi^*)(q, \psi^*)|$.

Suppose $(f \circ (\gamma_i)_B)^{-1}(0) \neq \emptyset$. By the length formula, there is an admissible curve $\tilde{\gamma}_i$ with endpoints $(p, \varphi), (q, \psi)$ that is not longer than γ_i , such that $(\tilde{\gamma}_i)_B = (\gamma_i)_B$ and $(\tilde{\gamma}_i)_F$ is constant on each maximal subinterval on which $f \circ (\gamma_i)_B > 0$. Thus we may assume $(\gamma_i)_F$ has this form, hence $\text{length } \gamma_i = \text{length}(\gamma_i)_B$. There are curves γ_i^* of the same form in W^* with endpoints $(p, \varphi^*), (q, \psi^*)$, such that $(\gamma_i^*)_B = (\gamma_i)_B$ and $\text{length } \gamma_i^* = \text{length}(\gamma_i)_B = \text{length } \gamma_i$.

Therefore in all cases, $|(p, \varphi)(q, \psi)| \geq |(p, \varphi^*)(q, \psi^*)|$. Reversing the roles of W and W^* proves (1).

To prove (2), suppose γ_F has length $L > |\varphi\psi|$. Set $v = \text{speed } \gamma_F$. Let $\beta : J \rightarrow F$ be a curve with endpoints φ, ψ and length $L' < L$. Then the length of γ is reduced by replacing γ_F with the reparametrization of β with speed $(L'/L) \cdot v$. This contradiction gives (2).

(3) is immediate from (1) and the length formula (4.1). \square

The two-piece property in the next proposition is worthy of note.

Proposition 4.6 (Vanishing warping function). *Let $\gamma = (\gamma_B, \gamma_F) : J \rightarrow B \times_f F$ be a geodesic joining (p, φ) and (q, ψ) . Suppose $J_0 \neq \emptyset$ where $J_0 = (f \circ \gamma_B)^{-1}(0)$.*

- (1) *The restriction of γ_F to any maximal subinterval J_i of $J - J_0$ is constant. If J_i has no common endpoint with J , the constant can be changed to any other point in F and the resulting curve will still be a geodesic in $B \times_f F$ with the same endpoints.*
 (2) *$|(p, \varphi)(q, \psi)| = \text{length } \gamma_B$.*
 (3) *(Two-piece property) γ_B consists of two geodesics of B that intersect on the maximal subinterval $[t_0, t_1]$ of J having endpoints in J_0 .*

Proof. By the length formula (4.1), any curve in B joining p and q and passing through $Z = f^{-1}(0)$ is the projection of a curve in $B \times_f F$ of the same length joining (p, φ) and (q, ψ) , and such that the projection to F on each interval J_i is constant. Claims (1) and (2) follow.

It follows also that γ_B minimizes length of curves in B from p to q that intersect Z . If $t_0 = t_1$ in claim (3), it follows that $\gamma_B|_{[0, t_0]}$ and $\gamma_B|_{[t_0, 1]}$ are geodesics and the claim holds. Suppose $t_0 < t_1$. Then $\gamma_B|_{[0, t_1]}$ is a geodesic, since otherwise by the triangle inequality there is a curve from p to q passing through $\gamma_B(t_1) \in Z$ that is shorter than γ_B . Similarly, $\gamma_B|_{[t_0, 1]}$ is a geodesic. \square

Clairaut's theorem on geodesics of a surface of revolution extends to the metric setting. The proof that the formulas hold almost everywhere (claim 1 in the proof below) is in [AB 98]. Here we prove the new result that the speed v_B exists and is continuous for all t ; when $(f \circ \gamma_B)^{-1}(0) = \emptyset$, the same holds for v_F ; and when $(f \circ \gamma_B)^{-1}(0) \neq \emptyset$, setting $\bar{v}_F \equiv 0$ give a continuous extension of v_F .

Theorem 4.7 (Clairaut's theorem). *Let $\gamma = (\gamma_B, \gamma_F) : J \rightarrow B \times_f F$ be a geodesic with speed a . Then v_B and an extension \bar{v}_F of v_F are defined and Lipschitz continuous for all $t \in J$, and there is a constant $c(\gamma)$ such that*

$$(4.2) \quad (f \circ \gamma_B)^2 \cdot \bar{v}_F = c(\gamma);$$

$$(4.3) \quad \text{if } f \circ \gamma_B > 0, \text{ then } v_B = \sqrt{a^2 - (c(\gamma)/f \circ \gamma_B)^2}.$$

Proof. Suppose $J_0 \neq \emptyset$ where $J_0 = (f \circ \gamma_B)^{-1}(0)$. By Proposition 4.6(1), $v_F : J - J_0 \rightarrow F$ satisfies $v_F = 0$. In this case, v_F may be extended to all of J by setting $\bar{v}_F \equiv 0$, and (4.2) holds with $c(\gamma) = 0$. Moreover, from the two-piece property we conclude that γ_B is a geodesic, with the only exception possible being a single break point when $t_0 = t_1$; in this case, v_B still exists with constant value a .

So suppose $f \circ \gamma_B > 0$.

Since γ_B and γ_F are Lipschitz, the speeds v_B and v_F are defined almost everywhere, and the Lebesgue integral of speed on an interval is arc-length.

1. (4.2) and (4.3) hold almost everywhere.

This claim is proved in [AB 98, Theorem 3.1].

2. v_F is defined and continuous on J and satisfies (4.2).

Set $c = c(\gamma)$. Since $c/(f \circ \gamma_B)^2$ is defined and continuous on J , v_F has a continuous extension \bar{v}_F to all of J by claim 1. It follows that the arc-length function $s(t)$ of γ_F is obtained by integrating the continuous function \bar{v}_F , and so $ds/dt = \bar{v}_F$. Since γ_F is a pregeodesic, ds/dt is the speed of γ_F , i.e. $v_F = ds/dt = \bar{v}_F$. The claim follows.

3. v_B is defined and continuous on J and satisfies (4.3).

It suffices to assume J is an open interval containing 0, and show that v_B is defined and continuous at $t = 0$.

Suppose speed $\gamma = 1$. Let $f(\gamma_B(0)) = b > 0$. For $\epsilon > 0$, consider a ball $\mathcal{B} = \mathcal{B}(\gamma_B(0), 2r) \subset f^{-1}(b - \epsilon, b + \epsilon)$. If $0 < s_1, s_2 < r$, then $\gamma_B|_{[-s_1, s_2]} \subset \mathcal{B}$.

Comparing the warped product metric with the Cartesian product metric on $B \times_{b+\epsilon} F$,

$$(4.4) \quad s_1 + s_2 < \sqrt{(|\gamma_B(-s_1) \gamma_B(s_2)|^2 + (b + \epsilon)^2 \cdot |\gamma_F(-s_1) \gamma_F(s_2)|^2)}.$$

Since (4.2) holds almost everywhere,

$$|\gamma_F(-s_1) \gamma_F(s_2)| = \int_{-s_1}^{s_2} \frac{c}{f(\gamma_B(s))^2} ds < (s_1 + s_2) \cdot \frac{c}{(b - \epsilon)^2}.$$

Hence by (4.4),

$$|\gamma_B(-s_1) \gamma_B(s_2)|^2 > (s_1 + s_2)^2 \cdot \left(1 - \frac{(b + \epsilon)^2 \cdot c^2}{(b - \epsilon)^4}\right).$$

Similarly, comparison with $B \times_{b-\epsilon} F$ gives an upper bound:

$$|\gamma_B(-s_1) \gamma_B(s_2)|^2 < (s_1 + s_2)^2 \cdot \left(1 - \frac{(b - \epsilon)^2 \cdot c^2}{(b + \epsilon)^4}\right).$$

Dividing by $(s_1 + s_2)^2$, and taking the limit first as $s_1, s_2 \rightarrow 0$, then as $\epsilon \rightarrow 0$, we obtain that v_B^2 exists and equals $1 - (c/b)^2$, the value needed for continuity.

By (4.2), if speed $\gamma = a$ then $c(\gamma) = a \cdot c(\tilde{\gamma})$ where $\tilde{\gamma}$ is a unit-speed reparametrization of γ . Hence (4.3).

4. v_B and v_F are Lipschitz continuous.

This claim follows from (4.2) and (4.3), since we assume f is locally Lipschitz. \square

Remark 4.8. The original formulation of (4.3) in [AB 98] states that any geodesic for which f is nonvanishing has a constant-speed reparametrization γ satisfying

$$\frac{1}{2} v_B^2 + \frac{1}{2 (f \circ \gamma_B)^2} = E.$$

almost everywhere. In this form, Clairaut's equation has a potential theoretic interpretation, where the constant E is called the total energy and the terms equated to E are the kinetic and potential energies.

Corollary 4.9 (Vertical geodesics). *Let $\gamma = (\gamma_B, \gamma_F) : [-s_0, s_0] \rightarrow B \times_f F$ be a geodesic with speed a , where $\gamma(-s_0)$ and $\gamma(s_0)$ lie in the same vertical leaf $\{p\} \times F$. Then*

- (1) *If $f \circ \gamma_B > 0$, then $\gamma_F(0)$ is the midpoint of γ_F .*
- (2) *If $f \circ \gamma_B > 0$, there is a geodesic $\tilde{\gamma} : [-s_0, s_0] \rightarrow B \times_f F$ with the same endpoints as γ that is symmetric about its midpoint, i.e., $\tilde{\gamma}_B(-s) = \tilde{\gamma}_B(s)$.*
- (3) *If $(f \circ \gamma_B)^{-1}(0) \neq \emptyset$, then $\gamma_B(0)$ is a nearest point to p of $Z = f^{-1}(0)$.*
- (4) *The minimum value of $f \circ \gamma_B$ is $(f \circ \gamma_B)(0) = c(\gamma)/a$.*
- (5) *If $f \circ \gamma_B > 0$, then the speed v_B of γ_B satisfies $v_B(s) = 0$ if and only if $(f \circ \gamma_B)(s) = c(\gamma)/a$.*

Proof. Suppose $f \circ \gamma_B > 0$.

It suffices to take $F = [-\ell/2, \ell/2]$, $\gamma_F(-s_0) = -\ell/2$, $\gamma_F(s_0) = \ell/2$, where $\ell = \text{length } \gamma_F$. Indeed, by Proposition 4.5 (3), γ_B remains unchanged by this substitution; and by (4.2), $c(\gamma)$ also remains unchanged.

Without loss of generality, the midpoint of γ_F is $\gamma_F(s_1)$ for $s_1 \in [0, s_0]$.

The reflection of F in 0 induces an isometric reflection of $B \times_f F$. Let $\tilde{\gamma} : [-s_0 + 2s_1, s_0] \rightarrow B \times_f F$ be defined in two halves, for which the second half coincides with γ on $[s_1, s_0]$ and first half traces in reverse the reflection of the first half on $[-s_0 + 2s_1, s_1]$. Then $\tilde{\gamma}$ has the same endpoints as γ ; both arcs have the same length, $a(s_0 - s_1)$, hence $\text{length } \tilde{\gamma} = 2a(s_0 - s_1) \leq 2as_0 = \text{length } \gamma$, with equality if and only if $s_1 = 0$. Since γ is a geodesic, $s_1 = 0$, hence (1) and (2).

Since $\check{v}_B(0) = 0$ where \check{v}_B is the speed of $\check{\gamma}_B$, we also have $v_B(0) = 0$. By (4.3), the vanishing set of v_B is also the set on which $f \circ \gamma_B$ takes its minimum value $c(\gamma)/a$, hence (4) and (5).

Alternatively, suppose $(f \circ \gamma_B)^{-1}(0) \neq \emptyset$. Since γ_B minimizes length of loops in B at p that intersect Z , we have (3). (4) is immediate since $c(\gamma) = 0$. \square

Remark 4.10. Let L denote length γ as defined by (4.1). Then $L = L_\Sigma$, for

$$(4.5) \quad L_\Sigma = \sup_{t_0 < \dots < t_n} \sum_{i=1}^n d_i,$$

where the supremum is taken over all partitions $t_0 < \dots < t_n$ of J . Here, letting \bar{t}_i be a minimum point of $(f \circ \gamma_B)|_{[t_{i-1}, t_i]}$,

$$d_i = \begin{cases} |\gamma(t_i) \gamma(t_{i-1})|_{B \times (f \circ \gamma_B)(\bar{t}_i) \cdot F} & \text{if } (f \circ \gamma_B)(\bar{t}_i) > 0, \\ |\gamma_B(t_i) \gamma_B(t_{i-1})|_B & \text{if } (f \circ \gamma_B)(\bar{t}_i) = 0, \end{cases}$$

where $B \times (f \circ \gamma_B)(\bar{t}_i) \cdot F$ denotes the Cartesian product of B with a scaling of F . The choice of \bar{t}_i ensures the formula is well-defined and any sequence of successively refined sums is nondecreasing. The proof that $L = L_\Sigma$ proceeds as in the classical case for length of an absolutely continuous curve ([G 53]).

Since we are assuming f is locally Lipschitz, it can be verified that the length induced by the warped product metric agrees with L .

It turns out that mere continuity of f is not sufficient for the warped product construction, even if $f > 0$. (This corrects a misstatement in [AB 98, AB 04].) For example, let B be the union of intervals $B_i = [0, 1/i]$ glued at 0, $f(0) = 1$, $(f|_{B_i})(1/i) = 1/i$, $F = [0, 1]$. For $\gamma(t) = (0, t)$, $0 \leq t \leq 1$, we have $L = 1$, while the infimum of lengths of admissible curves joining any two points of γ is 0.

When f is positive and locally Lipschitz, the original and metric topologies agree, and Definition 4.1 determines a length structure in the sense of [BBI 01].

5. BASE AND WARPING FUNCTION, CAT

Let $y = \text{sn}^\kappa$ be the function on \mathbf{R} satisfying

$$y'' + \kappa y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Theorem 5.1. (1) Suppose $X \in \text{CAT}^\kappa$ and $S \subset X$ is ϖ^κ -convex. Then

$$\text{sn}^\kappa \circ \text{dist}_S$$

is a sinusoidally κ -convex function on $\mathcal{B}(S, \varpi^\kappa/2)$.

(2) [Pr 91] Suppose $X \in \text{CBB}^\kappa$ and $\partial X \neq \emptyset$. Then

$$\text{sn}^\kappa \circ \text{dist}_{\partial X}$$

is a sinusoidally κ -concave function on X .

Proof. See [AB 96, §3], or for (2), see [Pt 06, Theorem 3.3.1]. \square

The next lemma will allow us often to restrict attention to warped products whose fibers are intervals.

Lemma 5.2. Let (B, f, F) be a WP-triple. Suppose $B \times_f F \in \text{CAT}^\kappa$ or $B \times_f F \in \text{CBB}^\kappa$ respectively.

- (1) Let $\beta : J \rightarrow F$ be a unit-speed geodesic. Then under the embedding

$$\text{id} \times \beta : B \times_f J \rightarrow B \times_f F,$$

the intrinsic and extrinsic metrics of $B \times_f J$ agree.

- (2) There is a nontrivial interval J such that $B \times_f J \in \text{CAT}^\kappa$ or $B \times_f J \in \text{CBB}^\kappa$ respectively.

Proof. By Proposition 4.5 (3) and Proposition 4.6, the map $\text{id} \times \beta$ preserves geodesics. Therefore (1) holds.

By Proposition 4.5 (2), F is ϖ^κ -geodesic if $B \times_f F \in \text{CAT}^\kappa$, and F is geodesic if $B \times_f F \in \text{CBB}^\kappa$. Therefore (2) follows from (1) and the assumption $F \neq \text{point}$. \square

Lemma 5.3. *If $W = B \times_f J$ is a geodesic space, where J is an interval with interior point 0, then the warping function $f : B \rightarrow \mathbf{R}_{\geq 0}$ satisfies*

$$f(p) = \lim_{\epsilon \rightarrow 0} \frac{(\text{sn}^\kappa \circ \text{dist}_{B \times \{0\}})((p, \epsilon))}{\epsilon}.$$

Proof. By Proposition 4.3 (2), for $p \in B$ the curve $\beta(t) = (p, t)$ in W is rectifiable, with arc-length parameter s satisfying $t = f(p)s$. Then the speed of β exists and equals $f(p)$ almost everywhere. By Proposition 4.5 (1) (taking $F = F^* = J$ and $p = q$), if two points of $\{p\} \times J$ have the same intrinsic distance in $\{p\} \times J$ then they have the same extrinsic distance in W . Thus the speed of β is constantly $f(p)$ everywhere, and we have

$$f(p) = \lim_{\epsilon \rightarrow 0} \frac{|(p, \epsilon) - (p, -\epsilon)|}{2\epsilon}.$$

A geodesic γ realizing the distance from (p, ϵ) to $B \times \{0\}$ has a symmetric extension, which is a geodesic between (p, ϵ) and $(p, -\epsilon)$ since γ cannot be shortened. Thus

$$(5.1) \quad f(p) = \lim_{\epsilon \rightarrow 0} \frac{\text{dist}_{B \times \{0\}}((p, \epsilon))}{\epsilon}.$$

Since $\text{sn}^{\kappa'}(0) = 1$, the lemma follows. \square

Theorem 5.4 (Theorem 2.2 (1)). *Let (B, f, F) be a WP-triple. Set $Z = f^{-1}(0)$. If $B \times_f F \in \text{CAT}^\kappa$, then $B \in \text{CAT}^\kappa$ and $f \in \check{\mathcal{C}}^\kappa$.*

Proof. Proposition 4.3 (1) implies $B \in \text{CAT}^\kappa$.

By Lemma 5.2 (2), we may assume F is a non-trivial interval $J = [-\theta_0, \theta_0]$. Since any two points at distance $< \varpi^\kappa$ in $B \times_f J$ are joined by a unique geodesic, Proposition 4.3 (1) implies that each horizontal leaf $B \times \{\epsilon\}$ is a ϖ^κ -convex subset of $B \times_f J$. By Theorem 5.1 (1), $\text{sn}^\kappa \circ \text{dist}_{B \times \{0\}}$ is sinusoidally κ -convex on the tubular neighborhood $\mathcal{B}(B \times \{0\}, \varpi^\kappa/2)$, and hence on a neighborhood of (p, ϵ) in $B \times \{\epsilon\}$ for ϵ sufficiently small. By Lemma 5.3 and Proposition 4.3 (1), $f \in \check{\mathcal{C}}^\kappa$. \square

6. BASE AND WARPING FUNCTION, CBB

Recall that we write $\overline{p} \in B \times_f F$ when $f(p) = 0$, where \overline{p} is the equivalence class $\{(p, \varphi) : \varphi \in F\}$.

The next lemma contains what we need in this paper about tangent cones of warped products.

Lemma 6.1. *Let (B, f, F) be a WP-triple, and J be a closed interval.*

- (1) Suppose $B \times_f F \in \text{CBB}^\kappa$. Then $B \in \text{CBB}^\kappa$.
 (2) Suppose $B \times_f F \in \text{CBB}^\kappa$. If $f(p) = 0$, then $d_p f$ is defined and

$$T_{\overline{p}}(B \times_f F) = T_p B \times_{d_p f} F,$$

$$\Sigma_{\overline{p}}(B \times_f F) = \Sigma_p B \times_{(d_p f|_{\Sigma_p B})} F.$$

- (3) Suppose $B \times_f J \in \text{CBB}^\kappa$. If $f(p) > 0$, then

$$T_{(p,\varphi)}(B \times_f J) = \begin{cases} T_p B \times \mathbf{R}_{\geq 0}, & \text{if } \varphi = \text{endpoint of } J, \\ T_p B \times \mathbf{R}, & \text{if } \varphi = \text{interior point of } J. \end{cases}$$

$$\Sigma_{(p,\varphi)}(B \times_f J) = \begin{cases} [0, \pi/2] \times_{\sin \circ \text{id}} \Sigma_p B, & \text{if } \varphi = \text{endpoint of } J, \\ [0, \pi] \times_{\sin \circ \text{id}} \Sigma_p B, & \text{if } \varphi = \text{interior point of } J. \end{cases}$$

Proof. (1) follows from Proposition 4.3 (1). When $\dim B = 1$, we additionally use $\text{diam}(B \times_f F) \leq \varpi^\kappa$.

- (2) By the arc-length formula (4.1),

$$\lambda(B \times_f F) = (\lambda B) \times_{\lambda(f \circ i_\lambda)} F,$$

where $i_\lambda : \lambda X \rightarrow X$ is the tautological map.

By Lemma 3.3,

$$(T_{\overline{p}}(B \times_f F), o_{\overline{p}}) = \lim_{\lambda \rightarrow \infty} ((\lambda B) \times_{\lambda(f \circ i_\lambda)} F, \overline{p}).$$

The existence of this limit implies that $(f \circ \alpha)^+(0)$ exists for every geodesic α of B with $\alpha(0) = p$. (2) follows.

- (3) We may also write

$$\lambda(B \times_f F) = (\lambda B) \times_{f \circ i_\lambda} (\lambda F).$$

Thus we obtain (3), e.g. when $f(p) > 0$ and φ is an interior point of J ,

$$T_{(p,\varphi)}(B \times_f J) = \lim_{\lambda \rightarrow \infty} ((\lambda B) \times_{f \circ i_\lambda} (\lambda J), (p, \varphi)) = T_p B \times (f(p)\mathbf{R}) = T_p B \times \mathbf{R}.$$

□

Theorem 6.2 (Theorem 2.3 (1)). *Let (B, f, F) be a WP-triple. Suppose $B \times_f F \in \text{CBB}^\kappa$. Then for $Z = f^{-1}(0)$:*

- (1) $B \in \text{CBB}^\kappa$,
- (2) $Z \subset \partial B$,
- (3) $f \in \widehat{\mathcal{C}}^\kappa$.

Proof. By Lemma 5.2.(2), we may assume F is a non-trivial interval $J = [-\theta_0, \theta_0]$.

1. $B \in \text{CBB}^\kappa$.

See Lemma 6.1 (1).

2. $f|_{(B - (Z \cup \partial B))} \in \widehat{\mathcal{C}}^\kappa$.

Set $W = B \times_f [0, \theta_0]$.

For any curve in $B \times_f J$ connecting two points of W , any maximal open segment not in W can be reflected into W , thus giving a curve of equal length in W . Hence intrinsic distance in W equals distance in $B \times_f J$. Therefore $W \in \text{CBB}^\kappa$.

Consider $p \in B - (Z \cup \partial B)$. Since $p \notin \partial B$, $\Sigma_p B$ is without boundary. Let $0 < \epsilon < \theta_0$. Since $\Sigma_{(p,\epsilon)} W$ is the spherical suspension of $\Sigma_p B$ by Lemma 6.1 (3), and

hence is without boundary, then (p, ϵ) is an interior point of W . Also $(p, 0) \in \partial W$, since $\Sigma_{(p,0)} W$ is the hemispherical suspension of $\Sigma_p B$.

There is $c > 0$ and a neighborhood U in $B - (Z \cup \partial B)$ of p such that if $q \in U$ and $0 < \epsilon < c$, the nearest point in ∂W to (q, ϵ) lies in $B \times \{0\}$. It follows, by Theorem 5.1 (2) applied to $X = W$, that for any geodesic α in U , the restriction of $\text{sn}^\kappa \circ \text{dist}_{B \times \{0\}}$ to the geodesic $\alpha \times \{\epsilon\}$ in $B \times \{\epsilon\}$ (necessarily also a geodesic in W) is sinusoidally κ -concave. Thus $f|_{(B - (Z \cup \partial B))} \in \widehat{\mathcal{C}}^\kappa$ by Lemma 5.3.

3. $Z \subset \partial B$.

The claim is true if $\dim B = 1$. In that case, B is either a circle, or a closed interval, i.e. a connected closed subset of \mathbf{R} . If $f(p) = 0$ for some $p \in B - \partial B$, then geodesics of $B \times_f J$ bifurcate, contradicting $B \times_f J \in \text{CBB}^\kappa$. Specifically, we can choose an isometric imbedding $\gamma_B : [b, a] \rightarrow B$, where $b < 0 < a$, $\gamma_B(0) = p$, and $f(\gamma_B(a)) > 0$. By Proposition 4.6 (1), we may define $\gamma_F(s)$ when $s \notin Z$ to be 0 for $b \leq s \leq 0$, and either 0 or θ_0 for $0 < s \leq a$.

Now choose $n > 1$, and assume the claim is true whenever $\dim B = n$.

Suppose $\dim B = n+1$, and $f(p) = 0$ for some $p \in B - \partial B$. We have $\Sigma_{\overline{p}}(B \times_f J) \in \text{CBB}^1$ [BGP 92]. By Lemma 6.1 (2),

$$\Sigma_{\overline{p}}(B \times_f J) = \Sigma_p B \times_{(\text{d}_p f|_{\Sigma_p B})} J,$$

where $\dim(\Sigma_p B) = n$ and $\partial(\Sigma_p B) = \emptyset$ since $p \notin \partial B$. By the induction hypothesis, $(\text{d}_p f|_{\Sigma_p B}) > 0$. Therefore by claim 2, $(\text{d}_p f|_{\Sigma_p B}) \in \widehat{\mathcal{C}}^1$. This is impossible since $(\text{d}_p f|_{\Sigma_p B})$ must take a minimum by compactness. In this case, some geodesic to the minimum point must extend as a quasigeodesic on which $\text{d}_p f|_{\Sigma_p B}$ becomes negative, a contradiction. Hence claim 3.

4. $f \in \widehat{\mathcal{C}}^\kappa$.

Suppose α is a geodesic of B . If α has no internal intersection with ∂B , then $f \circ \alpha \in \widehat{\mathcal{C}}^\kappa$ by claims 2 and 3. Otherwise, $\alpha \subset \partial B$. Let $\hat{\alpha}$ be a subsegment of α obtained by arbitrarily small shortening at either endpoint. Since $B \in \text{CBB}^\kappa$, $\hat{\alpha}$ is the unique geodesic between its endpoints. Any sequence of geodesics with endpoints in $B - \partial B$, and approaching the endpoints of $\hat{\alpha}$, must lie in $B - \partial B$ and converge to $\hat{\alpha}$. Therefore $f \circ \hat{\alpha}$ is sinusoidally κ -concave and hence so is $f \circ \alpha$, as claimed. \square

Definition 6.3. Suppose $B \times_f F \in \text{CBB}^\kappa$, where (B, f, F) is a WP-triple. Set $Z(f) = f^{-1}(0)$, where $Z(f) \subset \partial B$ by Theorem 6.2. Define $B^\dagger(f)$ to be the result of gluing two copies of B along $\text{closure}(\partial B - Z(f))$. Define $f^\dagger : B^\dagger(f) \rightarrow \mathbf{R}_{\geq 0}$ by $f^\dagger = f \circ (\Pi^\dagger(f))$ where $\Pi^\dagger(f) : B^\dagger(f) \rightarrow B$ is the tautological map.

Now we use Petrunin's incomplete-globalization theorem, Theorem 3.2, to prove the following partial-boundary gluing theorem. Since the gluing theorem may be accessed at the level of direction spaces by induction on dimension, the task is to show that it transmits to the underlying space.

Theorem 6.4 (Theorem 2.3 (2)). *Let (B, f, F) be a WP-triple. Suppose $B \times_f F \in \text{CBB}^\kappa$. Then $B^\dagger(f) \in \text{CBB}^\kappa$ and $f^\dagger \in \widehat{\mathcal{C}}^\kappa$.*

Proof. By Lemma 5.2 (2), we may assume F is a non-trivial interval $J = [-\theta_0, \theta_0]$.

Let us write $Z = Z(f) = f^{-1}(0)$, $B^\dagger = B^\dagger(f)$ and $\Pi^\dagger = \Pi^\dagger(f)$. By Theorem 6.2 (2), $Z \subset \partial B$.

For $a \geq 0$, set

$$B_a^\dagger = (f^\dagger)^{-1}((a, \infty)), \quad f_a^\dagger = f^\dagger|_{B_a^\dagger}.$$

Then $B_0^\dagger = (\Pi^\dagger)^{-1}(B - Z)$. Let $\overline{B_a^\dagger}$ be the closure of B_a^\dagger in B^\dagger .

1. $\text{curv } B_0^\dagger \geq \kappa$.

Let B^\ddagger denote the double of B . By Perelman's doubling theorem, $B^\ddagger \in \text{CBB}^\kappa$ [Pr 91, Theorem 5.2]. The natural embedding of the space B_0^\dagger in B^\ddagger is a local isometry. Hence the claim.

2. $f_0^\dagger \in \widehat{\mathcal{C}}^\kappa$.

Let W be the preimage of $B \times_f [0, \theta_0]$ under the tautological map

$$(B \times_f J)^\ddagger \rightarrow B \times_f J.$$

By reflection, (as in the proof of Theorem 6.2, Claim 2) intrinsic distance in W equals distance in $(B \times_f J)^\ddagger$. Since $(B \times_f J)^\ddagger \in \text{CBB}^\kappa$, then $W \in \text{CBB}^\kappa$.

For any $q \in B_0^\dagger$, there is $c > 0$ and a neighborhood U of q in B_0^\dagger such that if $p \in U$ and $0 < \epsilon < c$, the nearest point in ∂W to $(p, \epsilon) \in W$ lies in $B_0^\dagger \times \{0\}$. It follows, by Theorem 5.1 (2) applied to $X = W$, that for any geodesic α in U , the restriction of $\text{sn}^\kappa \circ \text{dist}_{B^\dagger \times \{0\}}$ to the geodesic $\alpha \times \{\epsilon\}$ in $B_0^\dagger \times \{\epsilon\}$ is sinusoidally κ -concave. By Lemma 5.3, the claim follows.

3. *The theorem holds if $\dim B = 1$.*

Since $B \in \text{CBB}^\kappa$, B is a circle of length $\leq 2 \cdot \varpi^\kappa$ or a closed interval of length $\leq \varpi^\kappa$. By Theorem 6.2 (2), $Z \subset \partial B$. If $B = \text{circle}$ or $Z = \partial B$, then $B^\dagger = B$ and the claim already holds by Theorem 6.2. So we may assume either B is a ray and $Z = \emptyset$, or B is a finite closed interval and $Z = \emptyset$ or an endpoint.

We have $f_0^\dagger \in \widehat{\mathcal{C}}^\kappa$ by claim 2. If B is a ray, then $\kappa \leq 0$, $B^\dagger \in \text{CBB}^\kappa$, and $f^\dagger = f_0^\dagger \in \widehat{\mathcal{C}}^\kappa$. If B^\dagger is a circle, then $B^\dagger \in \text{CBB}^\kappa$ and $f^\dagger = f_0^\dagger \in \widehat{\mathcal{C}}^\kappa$. If B^\dagger is an interval, then $f^\dagger \in \widehat{\mathcal{C}}^\kappa$. It follows that B^\dagger is an interval of length $\leq \varpi^\kappa$ and $B^\dagger \in \text{CBB}^\kappa$.

4. *Choose $n \geq 1$, and assume the theorem holds if $\dim B = n$. Suppose $\dim B = n + 1$. Suppose $\alpha^\dagger : I \rightarrow B^\dagger$ is a unit-speed geodesic of B^\dagger such that I is an interval with 0 in its interior, and $\alpha^\dagger(0) = p^\dagger$ where $(\Pi^\dagger)(p^\dagger) = p$ for some $p \in Z$. Then α^\dagger lies in $(f^\dagger)^{-1}(0) = (\Pi^\dagger)^{-1}Z$.*

It suffices to prove the claim for $I = (-\epsilon, \epsilon)$, for some $\epsilon > 0$.

Denote the gluing set by $G = \text{closure}(\partial B - Z)$. Then ∂B is the disjoint union

$$\partial B = \text{int } Z \sqcup G,$$

where int denotes interior relative to ∂B . The claim is clear if $p \in \text{int } Z$, so we assume $p \in G$.

We have $\Sigma_{\overline{p}}(B \times_f J) \in \text{CBB}^1$ [BGP 92]. By Lemma 6.1 (2),

$$\Sigma_{\overline{p}}(B \times_f J) = (\Sigma_p B) \times_{(d_p f|_{\Sigma_p B})} J,$$

where $\dim(\Sigma_p B) = n$. Set $(\Sigma_p B)^\dagger = (\Sigma_p B)^\dagger(d_p f|_{\Sigma_p B})$ and

$$\Pi_p^\dagger = \Pi^\dagger(d_p f|_{\Sigma_p B}) : (\Sigma_p B)^\dagger \rightarrow \Sigma_p B.$$

By the induction hypothesis,

$$(6.1) \quad (\Sigma_p B)^\dagger \in \text{CBB}^1 \text{ and } (d_p f|_{\Sigma_p B})^\dagger \in \widehat{\mathcal{C}}^1.$$

Then $(d_p f|_{\Sigma_p B})^\dagger$ is nonnegative and not identically 0, hence must take its minimum at a boundary point of $(\Sigma_p B)^\dagger$. (Otherwise there would be a quasigeodesic extension, along which $(d_p f|_{\Sigma_p B})^\dagger$ becomes negative, of a geodesic to a minimum point.)

Moreover, $(d_p f|_{\Sigma_p B})^\dagger$ has a unique maximum point u_{\max}^\dagger and

$$(6.2) \quad |u_{\max}^\dagger u^\dagger|_{(\Sigma_p B)^\dagger} \leq \pi/2$$

for any $u^\dagger \in (\Sigma_p B)^\dagger$. By uniqueness, if u_{\max} is the direction at which $(d_p f|_{\Sigma_p B}) \in \widehat{\mathcal{C}}^1$ takes its maximum, then $u_{\max} \in \partial(\Sigma_p B)$ and

$$(6.3) \quad u_{\max}^\dagger = (\Pi_p^\dagger)^{-1}(u_{\max}).$$

Let us write

$$B^\dagger = (B \times \{1, 2\}) / \sim$$

where $(q, 1) \sim (q, 2)$ if $q \in G$. Set $\alpha_1^\dagger = (\alpha^\dagger \circ (-\text{id}))|_{[0, \epsilon]}$ and $\alpha_2^\dagger = \alpha^\dagger|_{[0, \epsilon]}$.

Suppose both α_i^\dagger intersect $(\Pi^\dagger)^{-1}(G)$ only at $t = 0$. If both lie in $B \times \{1\}$, say, then α^\dagger lies in $\partial(B^\dagger)$ as desired. Thus we may suppose the α_i^\dagger lie in different copies of B .

Suppose α_i^\dagger intersects $(\Pi^\dagger)^{-1}(G)$ at some $t \neq 0$, for one or both i . Then we may shorten α_i^\dagger so that its endpoints lie on $(\Pi^\dagger)^{-1}(G)$. By reflecting maximal open segments in one copy of B into the other copy, we may obtain a curve with the same endpoints and length as α^\dagger and passing through p^\dagger , and which lies for $t \leq 0$ and $t \geq 0$ respectively in different copies of B .

Therefore we may assume that α_i^\dagger lies in $B \times \{i\}$.

Let $u_i^\dagger = (\alpha_i^\dagger)^+(0)$, $\alpha_i = \Pi^\dagger \circ \alpha_i^\dagger$, and $u_i = \alpha_i^+(0)$.

We may choose a geodesic direction $v \in \Sigma_p B$ arbitrarily close to u_{\max} . For a geodesic σ with $v = \sigma^+(0)$, let σ_i^\dagger be the geodesic in $B \times \{i\} \subset B^\dagger$ such that $\Pi^\dagger \circ \sigma_i^\dagger = \sigma$. Set $v_i^\dagger = (\sigma_i^\dagger)^+(0)$. Then

$$(6.4) \quad |u_i v|_{\Sigma_p B} = |u_i^\dagger v_i^\dagger|_{(\Sigma_p B)^\dagger},$$

since the righthand side is at most the lefthand side, and is not smaller by the reflection argument.

Now we are going to show

$$(6.5) \quad |u_1^\dagger u_{\max}^\dagger|_{(\Sigma_p B)^\dagger} + |u_{\max}^\dagger u_2^\dagger|_{(\Sigma_p B)^\dagger} = \pi.$$

Let us check that for any $\epsilon > 0$, if v is sufficiently close to u_{\max} then

$$|u_1 v|_{\Sigma_p B} + |v u_2|_{\Sigma_p B} \geq \pi - \epsilon.$$

Indeed, suppose not. By Lemma 3.3, there exists $0 < b < 1$ such that for some v arbitrarily close to u_{\max} ,

$$(6.6) \quad |\alpha_1(t) \sigma_1(b \cdot t)| + |\sigma_2(b \cdot t) \alpha_2(t)| \leq 2 \cdot a \cdot t + o(t),$$

where $a = \sin \frac{\pi - \epsilon}{2} < 1$. Moreover, since $(d_p f)(u_{\max}) > 0$, for $c = \frac{1-a}{3}$ and v sufficiently close to u_{\max} ,

$$(6.7) \quad \text{dist}_{(\partial B - Z)} \sigma(b \cdot t) \leq c \cdot t + o(t).$$

It follows from (6.6) and (6.7) that

$$|\alpha_1^\dagger(t) \alpha_2^\dagger(t)| \leq 2 \cdot (a + c) \cdot t + o(t),$$

where $a + c < 1$. Then a segment of the geodesic α^\dagger including p^\dagger does not minimize, a contradiction.

Therefore (6.5) follows from (6.3) and (6.4). By (6.2), each term on the lefthand side of (6.5) equals $\pi/2$. Then $(d_p f|_{\Sigma_p B})^\dagger(u_i^\dagger) = 0$ by (6.1). Thus $f \circ \alpha_i = 0$ by concavity of f , and α^\dagger lies in $(f^\dagger)^{-1}(0) = (\Pi^\dagger)^{-1}Z$ by Theorem 6.2 (2).

5. The theorem holds in all dimensions.

Choose $n \geq 1$, and assume the theorem holds if $\dim B = n$. Suppose $\dim B = n + 1$. By claim 4, B_0^\dagger is a geodesic space. By Theorem 3.2 and claim 1, $B^\dagger \in \text{CBB}^\kappa$. By claims 2 and 4, $f^\dagger \in \widehat{\mathcal{C}}^\kappa$. \square

Remark 6.5. In the proof of [AB 04, Theorem 6.2.2, case $\kappa \leq 0$], the following argument is outlined.

Given: a continuous function $\Phi(p, \theta) = f(p) \cos \theta : B \times_f J \rightarrow \mathbf{R}_{\geq 0}$ where $J = [-\pi/2, \pi/2]$, such that $\Phi \in \widehat{\mathcal{C}}^\kappa$, $\text{curv } \Phi^{-1}(0, \infty) \geq \kappa$, and $\kappa \leq 0$.

Prove: $B \times_f J \in \text{CBB}^\kappa$.

The argument suggested requires considerable preparation to fill in. There is a shorter proof by perturbation, similar to the perturbation argument in [AKP 08].

But Theorem 3.2 makes argument unnecessary. Since $\Phi \in \widehat{\mathcal{C}}^\kappa$, then $\Phi^{-1}((0, \infty))$ is convex in $B \times_f J$. Since $B \times_f J$ is the completion of $\Phi^{-1}((0, \infty))$, the claim follows immediately from Petrunin's incomplete-globalization theorem (Theorem 3.2).

7. CURVATURE OF THE FIBER, CBB

This section finishes the proof of Theorem 2.3, completing our consideration of curvature bounded below.

Theorem 7.1 (Theorem 2.3 (3) & (4)). *Suppose $B \times_f F \in \text{CBB}^\kappa$, where (B, f, F) is a WP-triple. Set $Z = f^{-1}(0)$.*

- (i) *If $Z = \emptyset$, then $\kappa \leq 0$, and $F \in \text{CBB}^{\kappa_F}$ for $\kappa_F = \kappa \cdot (\inf f)^2$.*
- (ii) *If $Z \neq \emptyset$, then $F \in \text{CBB}^{\kappa_F}$ for*

$$\begin{aligned} \kappa_F &= \sup \{ |\nabla_q f|^2 : q \in Z \} \\ &= \sup \{ (f \circ \alpha)^+(0)^2 : \alpha = \text{dist}_Z\text{-realizer with footpoint } \alpha(0) \in Z, |\alpha^+(0)| = 1 \}. \end{aligned}$$

Lemma 7.2. *If $B \times_f F \in \text{CBB}^\kappa$ for $\kappa \geq 0$, then one of these statements holds:*

- (a) $\emptyset \neq Z \subset \partial B$,
- (b) $f \equiv a > 0$, $Z = \emptyset$, $\kappa = 0$.

Proof. By Theorem 6.2 (2), $Z \subset \partial B$.

Let us write $B^\dagger = B^\dagger(f)$. By Theorem 6.4, we have $B^\dagger \in \text{CBB}^\kappa$ and $f^\dagger \in \widehat{\mathcal{C}}^\kappa$. Therefore along a quasigeodesic α^\dagger in the interior of B^\dagger , $f^\dagger \circ \alpha^\dagger$ is sinusoidally κ -concave, i.e. its value when $\kappa > 0$ (respectively, $\kappa = 0$) is supported from above by a multiple of a translate of sn^κ (respectively, by a linear function) having the same initial value and derivative. (The definition and properties of quasigeodesics are developed in [PP 94] and [Pt 06, Chapter 5]. Also see [AB 04, p. 1153] for a discussion of the support property used here.)

If $f \neq \text{constant}$, this derivative can be taken to be negative. Then α cannot be continued indefinitely in the interior of B^\dagger since $f^\dagger \circ \alpha^\dagger$ cannot become negative. Therefore α reaches ∂B^\dagger , where $f^\dagger = 0$. Hence $Z \neq \emptyset$.

If $f \equiv a > 0$, then since $f \in \widehat{\mathcal{C}}^\kappa$ for $\kappa \geq 0$, we must have $\kappa = 0$. \square

Proof of Theorem 7.1. The theorem is broken into three cases, which are proved in Propositions 7.3, 7.4, 7.6 below. \square

Proposition 7.3. *Suppose $B \times_f F \in \text{CBB}^\kappa$, where $\inf f > 0$. Then $F \in \text{CBB}^{\kappa_F}$ for $\kappa_F = \kappa \cdot (\inf f)^2$.*

Proof. By the length formula (4.1), F is closed in $B \times_f F$ and hence is complete.

Let us rescale f so that $\inf f = 1$, scaling the metric of F by the reciprocal factor so as to preserve W . Choose $p_i \in B$ such that $f(p_i) = 1 + a_i$ where $a_i \rightarrow 0$.

For $\varphi, \psi \in F$,

$$|\varphi \psi|_F \leq |(p_i, \varphi)(p_i, \psi)|_{B \times_f F} \leq |(p_i, \varphi)(p_i, \psi)|_{\{p_i\} \times F} = (1 + a_i)|\varphi \psi|_F,$$

where the first inequality is by the length formula (4.1).

Therefore

$$\lim_{i \rightarrow \infty} |(p_i, \varphi)(p_i, \psi)|_{B \times_f F} = |\varphi \psi|_F.$$

Since quadruples in $B \times_f F$ satisfy $(1 + 3)^\kappa$, so do quadruples in F . \square

Proposition 7.4. *Suppose $B \times_f F \in \text{CBB}^\kappa$, where $f > 0$ and $\inf f = 0$. Then $F \in \text{CBB}^0$.*

Proof. By Lemma 7.2, $\kappa < 0$.

Consider $p_i \in B$ such that $f(p_i) = a_i \rightarrow 0$. Set

$$\lambda_i = 1/a_i, \quad B_i = \lambda_i \cdot B, \quad f_i = \lambda_i \cdot f : B_i \rightarrow \mathbf{R}_{\geq 0}.$$

Then $B_i \times_{f_i} F \in \text{CBB}^{\kappa_i}$ where $\kappa_i = a_i^2 \cdot \kappa$. By Theorem 6.2, $B_i \in \text{CBB}^{\kappa_i}$, and $f_i \in \widehat{\mathcal{C}}^{\kappa_i}$.

Passing to a subsequence, we may assume that the pointed spaces (B_i, p_i) have Gromov-Hausdorff limit

$$\lim_{i \rightarrow \infty} (B_i, p_i) = (B_\infty, p_\infty).$$

1. For every $\epsilon > 0$ and $r > 0$, if i is sufficiently large then

$$1 - \epsilon < f_i | \overline{B}(p_i, r) < 1 + \epsilon.$$

Suppose α is a unit-speed geodesic of length $\leq r$ in B_i , with $\alpha(0) = p_i$. Extend B_i to its double B_i^\dagger , where $B_i^\dagger = B_i^\dagger(f_i)$ since $Z = \emptyset$. (If $\partial B = \emptyset$, then $B_i^\dagger = B_i$.) We extend f_i to f_i^\dagger , which in this case we write f_i^\dagger . Then $B_i^\dagger \in \text{CBB}^{\kappa_i}$, and $f_i^\dagger \in \widehat{\mathcal{C}}^{\kappa_i}$, by Perelman's doubling theorem and Theorem 6.4. In B_i^\dagger , we may extend α to $[0, \infty)$ as a quasigeodesic, on which $f_i^\dagger \circ \alpha$ satisfies the κ_i -concavity inequality. For any $q = \alpha(s_1)$, then $f_i^\dagger \circ \alpha$ is supported above by the κ_i -sinusoid that shares the same value and derivative at s_1 , i.e.

$$(7.1) \quad 0 < (f_i^\dagger \circ \alpha)(s) \leq f_i^\dagger(q) \cdot \cosh(\sqrt{-\kappa_i} \cdot (s - s_1)) + b \cdot \sinh(\sqrt{-\kappa_i} \cdot (s - s_1)),$$

where

$$(7.2) \quad b \cdot \sqrt{-\kappa_i} = (f_i^\dagger \circ \alpha)^+(s_1).$$

For $b \leq 0$, the exponential function

$$f_i^\dagger(q)e^{-\sqrt{-\kappa_i} \cdot (s-s_1)}$$

is the extreme possibility for such a supporting sinusoid that does not vanish on $[s_1, \infty)$. Therefore for any choice of α and any $s_1 \geq 0$

$$(7.3) \quad (f_i^\dagger \circ \alpha)^+(s_1) \geq -\sqrt{-\kappa_i} \cdot (f_i^\dagger \circ \alpha)(s_1)$$

(regardless of the sign of b). Integrating this differential inequality gives

$$(7.4) \quad (f_i^\dagger \circ \alpha)(s) \geq e^{-\sqrt{-\kappa_i} \cdot s}.$$

Now extend α to $(-\infty, \infty)$ as a quasigeodesic. Since f_i^\dagger is sinusoidally κ_i -concave along all of this extension, (7.3) also holds at $s_1 = 0$ for the left derivative:

$$(f_i^\dagger \circ \alpha)^-(0) \geq -\sqrt{-\kappa_i}.$$

The concavity property of $f_i^\dagger \circ \alpha$ tells us that the sum of its one-sided derivatives at any point is non-positive; hence $(f_i^\dagger \circ \alpha)^+(0) \leq -(f_i^\dagger \circ \alpha)^-(0) \leq \sqrt{-\kappa_i}$. Further, by (7.1) at $s_1 = 0$,

$$(f_i^\dagger \circ \alpha)(s) \leq \cosh(\sqrt{-\kappa_i} \cdot s) + b \cdot \sinh(\sqrt{-\kappa_i} \cdot s)$$

where $b \cdot \sqrt{-\kappa_i} = (f_i^\dagger \circ \alpha)^+(0)$ by (7.2). Therefore $b \leq 1$, and

$$(7.5) \quad (f_i^\dagger \circ \alpha)(s) \leq e^{\sqrt{-\kappa_i} \cdot s}.$$

By (7.4) and (7.5), if $0 < s \leq r$, and κ_i is sufficiently close to 0, then

$$1 - \epsilon \leq (f_i^\dagger \circ \alpha)(s) \leq 1 + \epsilon.$$

This proves claim 1.

Fixing ϵ and r and taking a limit as $i \rightarrow \infty$, and then letting $\epsilon \rightarrow 0$, gives $f_\infty = 1$ on the ball of radius r . Since r is arbitrary, $f_\infty = 1$ on B_∞ .

Now choose $\varphi_\infty \in F$. Set

$$W_i = \lambda_i \cdot (B \times_f F) = B_i \times_{f_i} F.$$

Then

$$(7.6) \quad \lim_{i \rightarrow \infty} (W_i, (p_i, \varphi_\infty)) = (W_\infty, (p_\infty, \varphi_\infty))$$

where $W_\infty = B_\infty \times F$. Indeed, we have $\lim_{i \rightarrow \infty} (B_i, p_i) = (B_\infty, p_\infty)$, i.e. given $r > 0$ and $\epsilon' > 0$, for all i sufficiently large there is $h_i : \mathcal{B}(p_i, r) \rightarrow B$ satisfying distortion $h_i < \epsilon'$, $f(p_i) = p_\infty$, and $\mathcal{B}(p_\infty, r - \epsilon') \subset \mathcal{B}(h_i(\mathcal{B}(p_i, r)), \epsilon')$. By claim 1, it follows that $H_i = h_i \times \text{id} : W_i \rightarrow W_\infty$ satisfies analogous conditions defining (7.6).

Since $W_i \in \text{CBB}^{\kappa_i}$, where $\kappa_i \rightarrow 0$, then $W_\infty \in \text{CBB}^0$. Hence $F \in \text{CBB}^0$. \square

Lemma 7.5. *Suppose $B \in \text{CBB}^\kappa$, and $f : B \rightarrow \mathbf{R}_{\geq 0}$ satisfies $f \in \widehat{\mathcal{C}}^\kappa$. Set $Z = f^{-1}(0)$ and suppose $\emptyset \neq Z \subsetneq \partial B$. If $B^\dagger(f) \in \text{CBB}^\kappa$ and $f^\dagger \in \widehat{\mathcal{C}}^\kappa$, then*

$$\begin{aligned} & \sup \{ |\nabla_q f|^2 : q \in Z \} \\ &= \sup \{ (f \circ \alpha)^+(0)^2 : \alpha = \text{dist}_Z\text{-realizer with footpoint } \alpha(0) \in Z, |\alpha^+(0)| = 1 \}. \end{aligned}$$

Proof. Let us write $B^\dagger = B^\dagger(f)$ and $\Pi^\dagger = \Pi^\dagger(f)$.

1. $Z = \text{closure}(\text{int } Z)$, where int denotes interior relative to ∂B .

Set $G = \text{closure}(\partial B - Z)$; then ∂B is the disjoint union

$$(7.7) \quad \partial B = \text{closure}(\text{int } Z) \sqcup \text{int } G.$$

This equation is purely topological, using only the duality of closure and int via complementation. The claim is an additional refinement, showing that when $B^\dagger \in \text{CBB}^\kappa$ and $f^\dagger \in \widehat{\mathcal{C}}^\kappa$, then gluing does not hide any vanishing points of f , but rather leaves all points of Z in $\partial(B^\dagger)$.

Suppose $p^\dagger \in B^\dagger$ satisfies $\Pi^\dagger(p^\dagger) = p \in Z$. Since $d_{p^\dagger}(f^\dagger) \geq 0$ and not identically 0, there is a geodesic α^\dagger in B^\dagger with $\alpha^\dagger(0) = p^\dagger$ and $(f^\dagger \circ \alpha^\dagger)^+(0) > 0$. If $p^\dagger \notin \partial(B^\dagger)$, there would be a quasigeodesic extension of α^\dagger across p^\dagger on which $f^\dagger \circ \alpha^\dagger < 0$, and this is impossible. Therefore

$$(7.8) \quad Z \cap \text{int } G = \emptyset.$$

The claim follows from (7.7) and (7.8).

2. Suppose $\alpha^\dagger : [0, \epsilon) \rightarrow B^\dagger$ is a unit-speed $\text{dist}_{\partial(B^\dagger)}$ -minimizer with footpoint $p^\dagger = \alpha^\dagger(0) \in \partial(B^\dagger)$. Then $\Sigma_{p^\dagger}(B^\dagger)$ is a hemispherical cone,

$$\Sigma_{p^\dagger}(B^\dagger) = [0, \pi/2] \times_{\sin \circ \text{id}} \partial(\Sigma_{p^\dagger}(B^\dagger)),$$

with vertex $u^\dagger = (\alpha^\dagger)^+(0)$. Moreover,

$$u^\dagger = |\nabla_{p^\dagger}(f^\dagger)|^{-1} \nabla_{p^\dagger}(f^\dagger),$$

and $h^\dagger = d_{p^\dagger}(f^\dagger) | \Sigma_{p^\dagger}(B^\dagger)$ satisfies

$$(7.9) \quad h^\dagger = h^\dagger(u^\dagger) \cdot (\cos \circ \text{dist}_{u^\dagger}).$$

The hemispherical cone structure of $\Sigma_{p^\dagger}(B^\dagger)$ at a footpoint in $\partial(B^\dagger)$ is derived in [Pr 91]; it is a direct corollary of the doubling theorem. The vertex of that cone is, as claimed, $u^\dagger = (\alpha^\dagger)^+(0)$.

We have $h^\dagger \in \widehat{\mathcal{C}}^1$ and $h^\dagger \geq 0$. Let v^\dagger be unique maximum point of h^\dagger . Let $\sigma^\dagger : [0, \ell] \rightarrow \Sigma_{p^\dagger}(B^\dagger)$ be a unit-speed local geodesic satisfying $\sigma^\dagger(0) = v^\dagger$ and passing through u^\dagger , extending to length $\ell \geq \pi/2$ before terminating at the equator. Then $(h^\dagger \circ \sigma^\dagger)(s) \leq h^\dagger(v^\dagger) \cdot \cos s$. Therefore $v^\dagger = u^\dagger$. (7.9) holds since $s = \text{dist}_{u^\dagger}(\sigma^\dagger(s))$.

3. Suppose $\alpha : [0, \epsilon) \rightarrow B$ is a unit-speed dist_Z -minimizer with footpoint $p = \alpha(0) \in Z$. Set $u = \alpha^+(0)$. Then $u = |\nabla_p f|^{-1} \nabla_p f$.

If $p \in \text{int } Z$, claim 3 follows from claim 2.

Suppose $p \notin \text{int } Z$. Let $\Pi^\dagger \circ \alpha^\dagger = \alpha$ and set $\alpha^\dagger(0) = p^\dagger$. By claim 1, $p^\dagger \in \partial(B^\dagger)$. By reflection, α^\dagger satisfies claim 2. Claim 3 follows.

4. Any $q \in Z$ is the limit of dist_Z -footpoints $p \in \text{int } Z$, i.e. points $p \in \text{int } Z$ such that p is the footpoint of a dist_Z -minimizer.

Any $q \in \text{int } Z$ is the limit of dist_Z -footpoints in $\text{int } Z$. Indeed, we may choose a curve $\alpha : [0, \epsilon) \rightarrow B$ with $\alpha(0) = q$ and $\alpha(t) \in B - \partial B$ for $t > 0$. For t sufficiently close to 0, $\alpha(t)$ has dist_Z -footpoints $p \in \text{int } Z$ arbitrarily close to q .

Therefore claim 4 follows from claim 1.

5. The lemma follows from claims 3 and 4.

It is straightforward to show (as in [Pt 06, Lemma 1.3.4]) that the function $|\nabla_q f|$ is lower semicontinuous on B , i.e. for any sequence $q_i \rightarrow q \in B$,

$$|\nabla_q f| \leq \liminf_{i \rightarrow \infty} |\nabla_{q_i} f|.$$

□

Proposition 7.6. *Suppose $B \times_f F \in \text{CBB}^\kappa$, where $Z \neq \emptyset$. Then $F \in \text{CBB}^{\kappa_F}$, where*

$$(7.10) \quad \begin{aligned} \kappa_F &= \sup \{ |\nabla_q f|^2 : q \in Z \} \\ &= \sup \{ (f \circ \alpha)^+(0)^2 : \alpha = \text{dist}_Z\text{-realizer with footpoint } \alpha(0) \in Z, |\alpha^+(0)| = 1 \}. \end{aligned}$$

Proof. By Lemma 7.5, it suffices to verify the first equality in (7.10).

1. *Proposition 7.6 holds for warped products with 1-dimensional base.*

In this case, B is isometric to a closed interval. If $p \in Z$, then p is an endpoint of B by Theorem 6.2 (2), so $\Sigma_p B = \{u\}$. By Lemma 6.1 (2),

$$\Sigma_{\overline{p}}(B \times_f F) = \{u\} \times_a F \cong a \cdot F,$$

where $a = d_p f(u) = |\nabla_p f|$. Therefore $a \cdot F \in \text{CBB}^1$, so $F \in \text{CBB}^{a^2}$.

2 (Induction step). *Suppose (7.10) holds for warped products with n -dimensional base. Then (7.10) holds for warped products with $(n+1)$ -dimensional base.*

Let $\dim B = n+1$.

Any $q \in Z$ is the limit of dist_Z -footpoints $p \in \text{int } Z$, by claim 4 of the proof of Lemma 7.5. By lower semicontinuity of $|\nabla_q f|$, it suffices to restrict the supremum in the first equality in (7.10) to dist_Z -footpoints $p \in \text{int } Z$.

Set $h = d_p f|_{\Sigma_p B}$. Then $\Sigma_p B \times_h F \in \text{CBB}^1$ by Lemma 6.1 (2). Since $\dim \Sigma_p B = n$, the induction hypothesis implies $F \in \text{CBB}^{\kappa_F}$ where

$$\kappa_F = \sup \{ |\nabla_v h|^2 : v \in \partial(\Sigma_p B) \}.$$

Since

$$|\nabla_v h| = |\nabla_p f|$$

for any $v \in \partial(\Sigma_p B)$ by (7.9), this completes the induction step. □

8. CURVATURE OF THE FIBER, CAT

This section finishes the proof of Theorem 2.2, completing our consideration of curvature bounded above.

In a Riemannian warped product $B \times_f F$, the vertical leaves $\{p\} \times F$ are umbilic, with extrinsic curvatures

$$|\nabla_p f|/f(p),$$

i.e. for a geodesic β in F , the curve (p, β) has curvature $|\nabla_p f|/f(p)$ at every point.

Since the acceleration of an intrinsic geodesic in a vertical leaf is towards the lower values of the warping function, the intuition behind this formula is that we actually need the downward gradient length $|\nabla_p(-f)|$, which however agrees with $|\nabla_p f|$ in Riemannian manifolds. This agreement need not occur in CAT spaces, so we expect the downward gradient to appear, as in Lemma 8.4.

In metric spaces, a theory of curvature of curves was developed in [AB 96]. Building on work of Lytchak [L 04], a ‘‘Gauss equation’’ for CAT^κ spaces was proved in [AB 06], i.e. a sharp upper curvature bound on a subspace whose intrinsic

geodesics have an extrinsic curvature bound. Now we are going to apply this work to obtain the correct curvature bound for the fiber in a CAT^κ warped product.

Definition 8.1 (Extrinsinc curvature). Suppose $Y \subset X$, where X is an intrinsic metric space and the intrinsic metric induced on Y is complete. Then Y is a subspace of *extrinsic curvature* $\leq A$, where $A \geq 0$, if intrinsic distances ρ in Y and extrinsic distances s in X satisfy

$$(8.1) \quad \rho - s \leq (A^2/24) \cdot s^3 + o(s^3)$$

on all pairs of points having ρ sufficiently small.

Remark 8.2. A Riemannian submanifold has extrinsic curvature $\leq A$ if and only if its second fundamental form II satisfies $|II| \leq A$.

Theorem 8.3 (Gauss equation [AB 06]). *Suppose $X \in \text{CAT}^\kappa$. Let $Y \subset X$ be a subspace of extrinsic curvature $\leq A$. Then $\text{curv } Y \leq \kappa + A^2$.*

In light of the Gauss equation, we need to establish a sharp bound on extrinsic curvature of a vertical leaf.

Lemma 8.4. *Let (B, f, F) be a WP-triple. Suppose $B \times_f F \in \text{CAT}^\kappa$, where $\kappa \leq 0$. If $f(p) > 0$ and $|\nabla_p(-f)| \neq 0$, then the vertical leaf $\{p\} \times F$ in $B \times_f F$ has extrinsic curvature*

$$\leq |\nabla_p(-f)|/f(p).$$

Lemma 8.5. *Let (B, f, F) be a WP-triple. Suppose $B \times_f F \in \text{CAT}^\kappa$. Let $\gamma = (\gamma_B, \gamma_F)$ be a (necessarily unique) geodesic of $B \times_f F$ with endpoints (p, φ) , $(p, \psi) \in \{p\} \times F$, where $|\langle (p, \varphi), (p, \psi) \rangle_{B \times_f F}| < \varpi^\kappa$. Then γ_F is the unique pregeodesic of F with endpoints φ, ψ .*

Proof. By Proposition 4.5 (2), γ_F is a pregeodesic of F joining φ and ψ . By Proposition 4.5 (3), for any pregeodesic β of F joining φ and ψ there is a geodesic of $B \times_f F$ with endpoints (p, φ) and (p, ψ) that projects to a monotonic reparametrization of β . Thus γ_F is unique because γ is unique. \square

Definition 8.6. For $a > 0$, set

$$(8.2) \quad \text{Cone}_a = \mathbf{R}_{\geq 0} \times_{a \cdot \text{id}} \mathbf{R}.$$

Proof of Lemma 8.4.

1. *It suffices to take $F = \mathbf{R}$, i.e. to show that the vertical leaf $\{p\} \times \mathbf{R}$ in $B \times_f \mathbf{R}$ has extrinsic curvature*

$$\leq |\nabla_p(-f)|/f(p).$$

By Lemma 8.5, F is a locally geodesic space. By Proposition 4.5 (1), vertical leaves are umbilic, i.e. if two points of $\{p\} \times F$ have the same intrinsic distance in $\{p\} \times F$, then they have the same extrinsic distance in $B \times_f F$. It follows that we need only verify the extrinsic curvature definition (8.1) for endpoint pairs lying on a single geodesic in $\{p\} \times F$.

Let $\beta : J \rightarrow F$ be a unit-speed geodesic. By Lemma 5.2 (1), under the embedding

$$\text{id} \times \beta : B \times_f J \rightarrow B \times_f F,$$

the intrinsic and extrinsic metrics of $B \times_f J$ agree. The claim follows.

2. Let $\gamma = (\gamma_B, \gamma_{\mathbf{R}}) : [-s_0, s_0] \rightarrow B \times_f \mathbf{R}$ be a geodesic with endpoints $(p, \pm\psi_0)$. Then $\gamma_B(s) = \gamma_B(-s)$, and the speed $v_{\mathbf{R}}$ of $\gamma_{\mathbf{R}}$ satisfies $v_{\mathbf{R}}(s) = v_{\mathbf{R}}(-s)$ for $s \in [0, s_0]$. (The lemma concerns limits as $\psi_0 \rightarrow 0$, and in its proof we will be taking ψ_0 sufficiently small.)

The claim follows from Corollary 4.9 (1) and uniqueness of γ .

3. The geodesic of $B \times_f \mathbf{R}$ joining $(p, -\psi_0)$ and (p, ψ_0) , $\psi_0 > 0$, does not lie in the vertical leaf $\{p\} \times \mathbf{R}$.

We are going to use the data from a cone geodesic γ to construct a shorter curve in $B \times_f \mathbf{R}$, specifically a curve whose projection to B runs back and forth along a geodesic pointing in a direction of decreasing f .

Suppose $0 < a < |\nabla_p(-f)|$, and set $r = f(p)/a$. Let

$$\gamma = (\gamma_{[0,r]}, \gamma_{\mathbf{R}}) : [-s_0, s_0] \rightarrow \text{Cone}_a$$

be a unit-speed geodesic with endpoints $(r, \pm\psi_0)$. Here we write $\gamma_{[0,r]}$ to emphasize that the projection of the cone geodesic γ to the base $\mathbf{R}_{\geq 0}$ of Cone_a lies in $[0, r]$.

Since the sector of Cone_a with angle at the vertex $2\psi_0$ is isometric to a sector of the Euclidean plane of the same angle, if $\psi_0 < \pi$ then γ may be viewed simply as a Euclidean segment connecting two points of a central circular arc. Hence its projection to the base behaves as described in claim 2 and has speed 0 only at $s = 0$.

We may choose a unit-speed geodesic $\alpha : [0, t_0] \rightarrow B$ with $\alpha(0) = p$ and such that

$$(f \circ \alpha)(t) < f(p) - a \cdot t.$$

For some $\epsilon > 0$, if $0 < \psi_0 < \epsilon$ then $r - \gamma_{[0,r]}(0) < t_0$. Then we may define a curve $\tilde{\gamma} : [-s_0, s_0] \rightarrow B \times_f \mathbf{R}$ by

$$\tilde{\gamma}(s) = (\alpha(r - \gamma_{[0,r]}(s)), \gamma_{\mathbf{R}}(s)).$$

By the length formula (4.1),

$$\text{length } \tilde{\gamma} < \text{length } \gamma < 2 \cdot f(p) \cdot \psi_0.$$

Since $2 \cdot f(p) \cdot \psi_0$ is the length of the geodesic joining $(p, -\psi_0)$ and (p, ψ_0) in the vertical leaf $\{p\} \times \mathbf{R}$, the claim follows.

4. Let $\gamma = (\gamma_B, \gamma_{\mathbf{R}}) : [-s_0, s_0] \rightarrow B \times_f \mathbf{R}$ be a unit-speed geodesic. Then:

- (i) The arclength parameter $t \in [-t_0, t_0]$ of γ_B is a strictly increasing function $t = t(s)$ of the arc-length parameter $s \in [-s_0, s_0]$ of γ .
- (ii) $(f \circ \gamma_B)(s(t))$ is a convex function of the arc length parameter t of γ_B .

(i) follows from claim 3.

If $(f \circ \gamma_B)(s(t))$ is not convex, then its restriction to some subinterval I of $[-t_0, t_0]$ is \geq the linear function of t with the same endpoint values. Moreover, $\gamma_B|I$ is not a geodesic since f is convex. Let α be the geodesic of B joining the endpoints of $\gamma_B|I$ and parametrized by I . Since α is shorter than $\gamma_B|I$, and $f \circ \alpha$ is convex, then the length formula shows that $\gamma|I$ can be shortened in $B \times_f \mathbf{R}$. This contradiction proves (ii).

5. Let $\gamma = (\gamma_B, \gamma_{\mathbf{R}}) : [-s_0, s_0] \rightarrow B \times_f \mathbf{R}$ be a unit-speed geodesic with endpoints $(p, \pm\psi_0)$. Then there exists $0 < A \leq |\nabla_p(-f)|/f(p)$ such that

$$(8.3) \quad 2 \cdot f(p) \cdot \psi_0 - 2 \cdot s_0 \leq (A^2/24) \cdot (2 \cdot s_0)^3 + o(s_0^3).$$

In contrast to the proof of claim 3, here we use the data from a geodesic in $B \times_f \mathbf{R}$ to construct a shorter curve in a cone.

As before, let $t \in [-t_0, t_0]$ be the arc-length parameter of γ_B . By Corollary 4.9(4) and (5), v_B vanishes only when $f \circ \gamma_B$ takes its minimum value, and in particular at $s = 0$. Moreover the minimum value occurs only at $s = 0$. Otherwise, since $(f \circ \gamma_B)(s(t))$ is convex by claim 4, then $(f \circ \gamma_B)(s(t))$ would take its minimum on a nontrivial interval I . By claim 2, I would be symmetric about 0. Then $\gamma_B|I$ would be constant, since otherwise γ could be shortened by replacing $\gamma_B|I$ with a constant curve. But $\gamma_B|I$ cannot be constant by claim 3.

For a given $a > 0$, we may reduce ψ_0 if necessary so that $t_0 < r = f(p)/a$. Define a curve $\tilde{\gamma} : [-s_0, s_0] \rightarrow \text{Cone}_a$ with endpoints $(r, \pm \psi_0)$, by requiring the projections $\tilde{\gamma}_{[0, r]}$ and $\tilde{\gamma}_{\mathbf{R}}$ of $\tilde{\gamma}$ on base and fiber to have speeds v_B and $v_{\mathbf{R}}$ respectively. By the length formula (4.1), if

$$f(p) - a \cdot t = a \cdot (r - t) \leq (f \circ \gamma_B)(s(t)), \quad 0 \leq t \leq t_0,$$

then $\text{length } \tilde{\gamma} \leq \text{length } \gamma$. In particular, appealing to claim 4, let us take

$$a = \left| \frac{d(f \circ \gamma_B)(s(t))}{dt}((-t_0)^+) \right| \leq |\nabla_p(-f)|.$$

Now we compare the respective curvatures of vertical leaves in $B \times_f \mathbf{R}$ and Cone_a . Since $2 \cdot f(p) \cdot \psi_0$ is the distance in the vertical leaf of $B \times_f \mathbf{R}$ between $(p, -\psi_0)$ and (p, ψ_0) , the curvature formula (8.1) gives

$$(8.4) \quad 2 \cdot s_0 = \text{length } \gamma \geq \text{length } \tilde{\gamma} \geq 2 \cdot f(p) \cdot \psi_0 - (A^2/24) \cdot (2 \cdot f(p) \cdot \psi_0)^3 + o(\psi_0^3),$$

where $A = a/f(p) = 1/r$ is the curvature of the fiber $\{t_0\} \times \mathbf{R}$ in Cone_a .

In the limit as $\psi_0 \rightarrow 0$, the inequality for $(2 \cdot f(p) \cdot \psi_0 - 2 \cdot s_0)$ given by (8.4) yields (8.3), proving the claim.

Lemma 8.4 follows from (8.3), claim 1 and Definition 8.1. \square

Lemma 8.7. *Suppose $B \in \text{CAT}^\kappa$, and $f : B \rightarrow \mathbf{R}_{\geq 0}$ satisfies $f \in \check{\mathcal{C}}^\kappa$. Set $Z = f^{-1}(0)$ and suppose $\emptyset \neq Z \subsetneq B$. Then:*

$$(8.5) \quad \inf \{ (f \circ \alpha)^+(0)^2 : \alpha = \text{dist}_Z\text{-realizer with footpoint } \alpha(0) \in Z, |\alpha^+(0)| = 1 \} \\ = \liminf_{\epsilon \rightarrow 0} \{ |\nabla_p(-f)|^2 : 0 < \text{dist}_Z(p) \leq \epsilon \}.$$

Proof. Let p satisfy $0 < \text{dist}_Z(p) \leq \epsilon$, and α be a geodesic realizing $\text{dist}_Z(p)$ with footpoint $\alpha(0) \in Z$, $|\alpha^+(0)| = 1$.

1. $(f \circ \alpha)^+(0)^2 \geq C_\epsilon^2$ where

$$C_\epsilon = \inf \{ |\nabla_q(-f)| : 0 < \text{dist}_Z(q) \leq \epsilon \}.$$

The claim is trivial if $C_\epsilon = 0$, so assume $C_\epsilon > 0$. Let η_t be the unit-speed downward gradient curve of f starting at $\alpha(t)$. Then for t sufficiently small, η_t remains within distance ϵ of Z since

$$(f \circ \eta_t)^+ = -|\nabla_{\eta_t}(-f)| \leq -C_\epsilon.$$

Indeed, if $t < \epsilon/2$ and $f(\alpha(t)) < C_\epsilon \cdot \epsilon/2$, then $f \circ \eta_t$ reaches 0 before $\text{dist}_Z \circ \eta_t$ can exceed ϵ .

Let $s(t) \geq t$ be the length of η_t . Then

$$(8.6) \quad f(\alpha(t)) = \int_{[0, s(t)]} |\nabla_{\eta_t(u)}(-f)| du \geq s(t) \cdot C_\epsilon \geq t \cdot C_\epsilon.$$

The claim follows.

$$2. (f \circ \alpha)^+(0)^2 \leq |\nabla_p(-f)|^2 / \cos^2 \epsilon.$$

We may assume for this claim that $\kappa = 1$, since scaling changes both sides of the inequality by the same positive factor.

By $(f \circ \alpha)'$ we mean the left-sided or the right-sided derivative, which are equal a.e. by semiconvexity of f . For the same reason, $(f \circ \alpha)'$ is continuous except for countably many upward jumps. When $(f \circ \alpha)' > 0$, the sinusoidal 1-convexity of f implies that $[(f \circ \alpha)']^2 + (f \circ \alpha)^2$ is nondecreasing.

Let us abbreviate $f \circ \alpha$ by f . Suppose $p = \alpha(t_0)$; then $t_0 \leq \epsilon$. For $t_0 < \pi$, the 2-point sine curve bounding f above on $[0, \bar{t}]$, $0 < \bar{t} < t_0$, is

$$\frac{f(\bar{t})}{\sin \bar{t}} \cdot \sin t.$$

Then

$$f'(\bar{t}) \geq \frac{f(\bar{t})}{\sin \bar{t}} \cdot \cos \bar{t}, \quad \text{that is, } f(\bar{t}) \leq \frac{\sin \bar{t}}{\cos \bar{t}} \cdot f'(\bar{t}).$$

In particular, $f'(\bar{t}) > 0$ for $t_0 < \pi/2$.

The claim follows:

$$\begin{aligned} f^+(0)^2 &\leq \lim_{\bar{t} \rightarrow t_0} (f'(\bar{t})^2 + f(\bar{t})^2) \\ &\leq \lim_{\bar{t} \rightarrow t_0} (f'(\bar{t})^2 / \cos^2 \bar{t}) \\ &\leq |\nabla_p(-f)|^2 / \cos^2 t_0. \end{aligned}$$

The lemma follows from Claims 1 and 2. \square

Theorem 8.8 (Theorem 2.2 (2) & (3)). *Suppose $B \times_f F \in \text{CAT}^\kappa$, where (B, f, F) is a WP-triple. Set $Z = f^{-1}(0)$.*

(i) *If $Z = \emptyset$, then $F \in \text{CAT}^{\kappa_F}$ for $\kappa_F = \kappa \cdot (\inf f)^2$.*

(ii) *If $Z \neq \emptyset$, then $F \in \text{CAT}^{\kappa_F}$ for $\kappa_F = \min \{\kappa_{foot}, \kappa_{far}\}$, where*

$$\kappa_{foot} = \inf \{(f \circ \alpha)^+(0)^2 : \alpha = \text{dist}_Z\text{-realizer with footpoint } \alpha(0) \in Z, |\alpha^+(0)| = 1\}$$

$$= \liminf_{\epsilon \rightarrow 0} \{|\nabla_p(-f)|^2 : 0 < \text{dist}_Z(p) \leq \epsilon\},$$

$$\kappa_{far} = \inf \{\kappa \cdot f(p)^2 : \text{dist}_Z(p) \geq \varpi^\kappa / 2\}.$$

Proof.

1. *Suppose $\kappa \leq 0$. If $f(p) > 0$, then*

$$\text{curv } F \leq \kappa \cdot f(p)^2 + |\nabla_p(-f)|^2.$$

If $|\nabla_p(-f)| = 0$, then p is a minimum point of f since f is convex. By Proposition 4.3 (3), the intrinsic and extrinsic metrics of $\{p\} \times F$ agree. Hence $\{p\} \times F \in \text{CAT}^\kappa$, and the claim follows by scaling.

Suppose $|\nabla_p(-f)| > 0$. By Lemma 8.4 and Theorem 8.3,

$$\text{curv}(\{p\} \times F) \leq (\kappa \cdot f(p)^2 + |\nabla_p(-f)|^2) / f(p)^2.$$

Hence the claim.

2. *Suppose $\kappa \leq 0$. If $Z \neq \emptyset$, then*

$$\text{curv } F \leq \liminf_{\epsilon \rightarrow 0} \{|\nabla_p(-f)|^2 : 0 < \text{dist}_Z(p) \leq \epsilon\}.$$

The claim follows from claim 1.

3. Suppose $\kappa \leq 0$. If $Z = \emptyset$, then $\text{curv } F \leq \kappa \cdot (\inf f)^2$.

Consider the sublevel sets

$$S_i = \{p \in B : f(p) \leq (\inf f) + 1/i\}.$$

Choose $C > 0$. If i is sufficiently large, there is some $p_i \in S_i$ such that $|\nabla_{p_i}(-f)| \leq C$. Indeed, suppose not. If η is a downward gradient curve of f starting at a point of S_i , then $f \circ \eta$ must take values $\leq \inf f$, a contradiction. Therefore this claim follows from claim 1.

4. Let κ be arbitrary.

(i) If $Z = \emptyset$, then $\text{curv } F \leq \kappa_F$ for $\kappa_F = \kappa \cdot (\inf f)^2$,

(ii) If $Z \neq \emptyset$, then $\text{curv } F \leq \kappa_F$ for $\kappa_F = \min\{\kappa_{\text{foot}}, \kappa_{\text{far}}\}$, where

$$\kappa_{\text{foot}} = \inf\{(f \circ \alpha)^+(0)^2 : \alpha = \text{dist}_Z\text{-realizer with footpoint } \alpha(0) \in Z, |\alpha^+(0)| = 1\}$$

$$\kappa_{\text{far}} = \inf\{\kappa \cdot f(p)^2 : \text{dist}_Z(p) \geq \varpi^\kappa/2\}.$$

By claims 2 and 3 and Lemma 8.7, the claim is true if $\kappa \leq 0$. So assume $\kappa > 0$, and without loss of generality, set $\kappa = 1$.

Let $\text{Cone } f$ be the homogeneous linear extension of f to $\text{Cone } B$. Then

$$\begin{aligned} \text{Cone}(B \times_f F) &= \mathbf{R}_{\geq 0} \times_{\text{id}} (B \times_f F) \\ &= (\mathbf{R}_{\geq 0} \times_{\text{id}} B) \times_{\text{Cone } f} F = \text{Cone } B \times_{\text{Cone } f} F. \end{aligned}$$

We have $\text{Cone}(B \times_f F) \in \text{CAT}^0$ (see Remark 2.5 (b)). Moreover, since $f \in \check{\mathcal{C}}^1$, then $\text{Cone } f \in \check{\mathcal{C}}^0$ (see [AB 96, Lemma 3.5]). We also have

$$Z(\text{Cone } f) = \text{Cone } Z(f) \cup o \neq \emptyset.$$

Suppose $\check{\alpha} : [0, a] \rightarrow \text{Cone } B$ is a geodesic realizing $\text{dist}_{Z(\text{Cone } f)}(p, 1)$ for some $p \in B$, with footpoint $\check{\alpha}(0) \in Z(\text{Cone } f)$ and $|\check{\alpha}^+(0)| = 1$. By claim 2 and Lemma 8.7,

$$(8.7) \quad \text{curv } F \leq ((\text{Cone } f) \circ \check{\alpha})^+(0)^2.$$

Suppose $Z = Z(f) = \emptyset$. Then $Z(\text{Cone } f) = \{o\}$, $\check{\alpha}(0) = o$, and $\check{\alpha}$ is the radial segment from o to $(p, 1)$. Thus $((\text{Cone } f) \circ \check{\alpha})^+(0) = f(p)$. (i) follows from (8.7) by rescaling.

On the other hand, suppose $Z = Z(f) \neq \emptyset$.

If $\text{dist}_Z(p) \geq \pi/2$, we again have $\check{\alpha}(0) = o$, so $\check{\alpha}$ is the radial segment from o to $(p, 1)$. In this case, $((\text{Cone } f) \circ \check{\alpha})^+(0) = f(p)$ and $\text{curv } F \leq \kappa \cdot f(p)^2$ by (8.7) and rescaling. Therefore $\text{curv } F \leq \kappa_{\text{far}}$.

Suppose $\text{dist}_Z(p) < \pi/2$. Let α be a geodesic in B realizing $\text{dist}_Z(p)$, with footpoint $\alpha(0) \in Z$, $|\alpha^+(0)| = 1$. Let $\text{Cone } \alpha$ be the cone over the image of α . If α has arclength parameter θ , then the intrinsic and extrinsic metrics of $\text{Cone } \alpha$ agree and are isometric to a sector of \mathbf{E}^2 with polar coordinates (r, θ) . In these coordinates, $(\text{Cone } f)|_{\text{Cone } \alpha} = r \cdot f(\theta)$. We choose $\check{\alpha}$ to lie in $\text{Cone } \alpha$, projecting to a reparametrization of α . (It is true that α and $\check{\alpha}$ are uniquely determined by p , but we do not use this fact.) A simple calculation in polar coordinates gives

$$((\text{Cone } f) \circ \check{\alpha})^+(0) = (f \circ \alpha)^+(0).$$

Thus (ii) follows from (8.7) and rescaling.

5. For ϵ sufficiently small, any $\varphi, \psi \in F$ such that $|\varphi\psi|_F < \varpi^{\kappa_F + \epsilon}$ are joined by a unique geodesic of F , and these geodesics depend continuously on φ, ψ .

Let κ_F be as in claim 4. It follows from claim 4 that either

- (a) $\kappa \leq 0$, or
- (b) $\kappa > 0$ and there is $p \in B - Z$ such that $\kappa_F + \epsilon \geq \kappa \cdot f(p)^2$.

Indeed, in case (b), if $Z \neq \emptyset$ then $\kappa_F \geq 0$ and $\kappa \cdot f(p)$ may be taken to be positive and arbitrarily close to 0.

In case (a), $B \times_f F \in \text{CAT}^0$, and so $(p, \varphi), (p, \psi) \in \{p\} \times F$ are joined by a geodesic γ in $B \times_f F$ that depends uniquely and continuously on its endpoints.

In case (b),

$$\varpi^{\tilde{\kappa}} \leq \varpi^{\kappa}, \quad \text{where } \tilde{\kappa} = \frac{\kappa_F + \epsilon}{f(p)^2}.$$

Therefore if $|(p, \varphi)(\psi, p)|_{\{p\} \times F} < \varpi^{\tilde{\kappa}}$, then $(p, \varphi), (p, \psi)$ are joined by a geodesic γ of length $< \varpi^{\kappa}$ in $B \times_f F$. Since $B \times_f F \in \text{CAT}^{\kappa}$, then γ depends uniquely and continuously on its endpoints.

Now the claim follows from Lemma 8.5.

6. $F \in \text{CAT}^{\kappa_F}$ where κ_F is defined in (i) and (ii).

The claim follows from claims 4 and 5, completeness of F , and Alexandrov's patchwork globalization theorem (see [BH 99, Proposition II.4.9] or [AKP, *Definitions of CBA*]). \square

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1409 W. GREEN ST., URBANA, ILLINOIS 61801
E-mail address: `sba@math.uiuc.edu`

1409 W. GREEN ST., URBANA, ILLINOIS 61801
E-mail address: `bishop@math.uiuc.edu`